Exercise 1. (**Markov Chain in a library**) In a library with n books, the ith book has probability p_i to be chosen at each request. To make it quicker to find the book the next time, the librarian moves the book to the left end of the shelf. Define the state of a Markov chain at any time to be the list of books we see as we examine the shelf from left to right. Since all the books are distinct, the state space E is the set of all permutations of the set $\{1, 2, \ldots, n\}$. Show that

$$\pi(i_1, \cdots i_n) = p_{i_1} \cdot \frac{p_{i_2}}{1 - p_{i_1}} \cdots \frac{p_{i_n}}{1 - p_{i_1} - \cdots p_{i_{n-1}}}$$

is a stationary distribution.

Solution. The distribution π is stationary for the Markov chain if and only if, supposing that there exists $n > 0$ such that $X_n \sim \pi$, X_{n+1} has also the same distribution π.
Suppose that X_n is distributed according to π. Notice that if we are in state $(i_1, \cdots, i_n)$ at time $n + 1$, then the only possibilities for the chain at time n are

$$S = \{(i_1, \cdots, i_n), \ \ (i_2, i_1, i_3 \cdots, i_n), \ \ (i_2, i_3, i_1, i_4, \cdots, i_n), \ \ \cdots \ (i_2, i_3, i_4, \cdots, i_n, i_1)\}. \tag{1}$$

Hence, to reach $(i_1, \cdots, i_n)$ at time $n+1$, we choose the book i_1 at time n (starting from one of the states in S given by (1)). We get

$$\begin{aligned}
\mathbb{P}(X_{n+1} = (i_1, \cdots, i_n)) &= \sum_{v \in S} \mathbb{P}(X_n = v)\mathbb{P}(X_{n+1} = (i_1, \cdots, i_n) \mid X_n = v) \qquad (2)\\
&= p_{i_1} \left(\pi(i_1, \cdots, i_n) + \pi(i_2, i_1, i_3, \cdots, i_n) + \cdots + \pi(i_2, i_3, i_4, \cdots, i_n, i_1)\right),
\end{aligned}$$

where we supposed that X_n is distributed according to π. It remains to show that the right term of (2) is equal to $\pi(i_1, \cdots, i_n)$. To simplify the computations, we suppose that $n = 3$. We then obtain, noticing that $1 - p_{i_1} - p_{i_2} = p_{i_3}$,

$$\begin{aligned}
p_{i_1} \frac{p_{i_2}}{1 - p_{i_1}} &\stackrel{?}{=} p_{i_1} \cdot (p_{i_1} \frac{p_{i_2}}{1 - p_{i_1}} + p_{i_2} \frac{p_{i_1}}{1 - p_{i_2}} + p_{i_2} \frac{p_{i_3}}{1 - p_{i_2}}) && \iff \\
p_{i_1} \frac{p_{i_2}}{1 - p_{i_1}} \cdot (1 - p_{i_1}) &\stackrel{?}{=} p_{i_1} \cdot (p_{i_2} \frac{p_{i_1}}{1 - p_{i_2}} + p_{i_2} \frac{p_{i_3}}{1 - p_{i_2}}) && \iff \\
p_{i_1} p_{i_2} (1 - \frac{p_{i_1}}{1 - p_{i_2}}) &\stackrel{?}{=} \frac{p_{i_1} p_{i_2} p_{i_3}}{1 - p_{i_2}} && \iff \\
\frac{p_{i_1} p_{i_2} p_{i_3}}{1 - p_{i_2}} &= \frac{p_{i_1} p_{i_2} p_{i_3}}{1 - p_{i_2}}.
\end{aligned}$$

Since the last equality is always verified, we deduce that all the others are also verified, and thus $\mathbb{P}(X_{n+1} = (i_1, i_2, i_3)) = \pi(i_1, i_2, i_3)$. Since the state (i_1, i_2, i_3) is random, we deduce that X_{n+1} is distributed according to π.
The argument can be easily generalized to any n, and hence π is the stationary distribution of the system.

Exercise 2. (**Random walk on a graph**)
An undirected graph $\mathcal{G}$ is a countable collection of states (that we call vertices) along with some edges connecting them. The degree d_i of a vertex i is the number of edges incident to i. We suppose the graph to be locally finite (i.e., each edge is incident to a finite number of

edges). We say that a Markov chain on the state space $E = \mathcal{G}$ is a random walk on the graph if the transition probabilities are given by

$$p_{i,j} = \begin{cases} 1/d_i & \text{if } (i,j) \text{ is an edge,} \\ 0 & \text{otherwise,} \end{cases}$$

for $i, j \in \mathcal{G}$.

a) We assume that $\mathcal{G}$ is connected (implying that P is irreducible) and that $\sum_i d_i < \infty$. Find the stationary distribution of the random walk on $\mathcal{G}$.
Hint: Assume that the random walk is reversible and find a stationary distribution verifying the detailed balance equations. Explain why P is reversible.

b) We assume now that the graph is a chessboard, i.e., the vertices are $\mathcal{G} = \{1, \dots, 8\}^2$ and the edges are the possible moves of a King. We assume that the King starts its random walk in one of the four corners of the chessboard $c \in \mathcal{G}$. Compute the mean return time to the initial state $\mathbb{E}_c(T_c)$ of the King. Compute the same quantity for a Knight instead of a King.

Solution. a) We start from the detailed balance equations:

$$\pi_i P_{ij} = \pi_j P_{ji}, \ \forall i, j \in \mathcal{G}. \tag{3}$$

This is equivalent to

$$\pi_i \cdot \frac{\mathbb{1}_{\{i \sim j\}}}{d_i} = \pi_j \cdot \frac{\mathbb{1}_{\{j \sim i\}}}{d_j},$$

where $\{i \sim j\}$ indicate the event where i and j are connected by an edge. If $i \sim j$, we have

$$\frac{\pi_i}{d_i} = \frac{\pi_j}{d_j}. \tag{4}$$

Otherwise, since the graph is connected, there exists a path $k_0 = i \sim k_1 \sim \cdots \sim k_m = j$ joining i and j. For all couple of points k_r, k_{r+1} the equality (4) is verified (for π_{k_r} et $\pi_{k_{r+1}}$). We deduce then that (4) is verified for all i and j of the graph. Using that $\sum_{i \in \mathcal{G}} \pi_i = 1$, we get easily that $\pi_i = \frac{d_i}{\sum_{j \in \mathcal{G}} d_j}$.

b) By a theorem seen in the course, we know that $\mathbb{E}_c[T_c] = \frac{1}{\pi_c} = \frac{\sum_{j \in \mathcal{G}} d_j}{d_c}$. It remains to find the degrees of all the vertices of the graph. If we are in one of the four corners, the degree (= the number of possible moves of the King) is equal to 3. If we are on one of the 24 boxes at the border of the chessboard and which is not one of the 4 corners, the degree of one of the box is 5. In all remaining states (for the 64-4-24=36 boxes remaining), the degree is 8. We get then:

$$\mathbb{E}_c[T_c] = \frac{\sum_{j \in \mathcal{G}} d_j}{d_c} = \frac{4 \cdot 3 + 24 \cdot 5 + 8 \cdot 36}{3} = 140.$$

Starting from a corner c, we need in average 140 moves of the King to come back to c.
The random walk of the Knight: counting the number of possible moves of the Knight starting from one of the 64 vertices of the graph, one gets the following configuration for the degrees:

2	3	4	4	4	4	3	2
3	4	6	6	6	6	4	3
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
3	4	6	6	6	6	4	3
2	3	4	4	4	4	3	2

Starting from one of the corners c, the expected time before return to c is given by:

$$\mathbb{E}_c[T_c] = \frac{1}{\pi_c} = \frac{\sum_{j \in \mathcal{G}} d_j}{d_c} = \frac{2 \cdot 4 + 3 \cdot 8 + 4 \cdot 20 + 16 \cdot 6 + 16 \cdot 8}{2} = 168.$$

Exercise 3. Let P be a transition matrix on a finite state space E.

(a) Prove the following linear algebra result: Given a matrix Q, Q and Q^t have the same eigenvalues.
Use this to prove that P has a stationary distribution π(i.e. a probability measure $\pi P = \pi$).

(b) Find an example, when E is an infinite state space, for which P doesn't have any stationary distribution.

Solution. (a) A real number λ is an eigenvalue for a matrix Q if and only if $\det(Q - \lambda I) = 0$ where I is the identity matrix of the same size as Q. We can write:

$$\det(Q^t - \lambda I) = \det(Q^t - \lambda I^t) = \det\left((Q - \lambda I)^t\right) = \det(Q - \lambda I).$$

This shows that Q and Q^t have the same eigenvalues.
Since P is a transition matrix, we have

$$P\mathbf{1} = \mathbf{1},$$

where $\mathbf{1}$ is a column vector of 1's of size $|E|$. Therefore, $\lambda = 1$ is an eigenvalue for P. By the above result, we know that there exists a vector $\mathbf{v}$ such that

$$P^t\mathbf{v} = \mathbf{v} \Longleftrightarrow \mathbf{v^t}P = \mathbf{v^t}.$$

Remark:
Using the Perron-Frobenius theorem (not seen in this course), we can choose the eigenvector $\mathbf{v} = (v_1, v_2, \cdots)$ such that $v_i \geq 0$ for all i.
The vector $\pi := \mathbf{v^t}/(\|\mathbf{v^t}\|)$ is a stationary distribution for P.

(b) If E is of infinite dimension, we can find examples for which $\|\mathbf{v^t}\| = \infty$ which implies $\pi \equiv \mathbf{0}$ and so is not a distribution anymore. We can, for example, consider the Markov chain in exercise 2a) in Serie 3. In this case we have

$$\pi_i = (\pi P)_i = \sum_j \pi_j P_{ji} = q\pi_i + p\pi_{i+1}.$$

This implies that $\pi_i = \pi_{i+1}$ for all $i \geq 0$ and so $\sum_i \pi_i = \infty$. In other words, π is not a stationary distribution.

Exercise 4. Let X be a Markov chain on E (not necessarily irreducible). Suppose that state $j \in E$ is positive recurrent and aperiodic. Show that

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j \mathbb{P}(\tau_j < \infty \mid X_0 = i), \qquad \tau_j = \inf\{n \geq 0 : X_n = j\},$$

where π is the stationary distribution of the chain restricted to the communicating class of j.

Solution. By the total probability formula, we have

$$p_{ij}^{(n)} = \mathbb{P}(X_n = j \mid X_0 = i) = \sum_{r=1}^{n} \mathbb{P}(\tau_j = r \mid X_0 = i) p_{jj}^{(n-r)}.$$

By considering the sub-chain with valued in the communicating class of j, we see that this sub-chain is irreducible and j is positive recurrent. We can thus apply a theorem from the class that gives that $p_{jj}^{(n-r)} \to \pi_j$. Since $\sum_{r=1}^{n} \mathbb{P}(\tau_j = r \mid X_0 = i) < \infty$, we obtain, by the dominated convergence theorem

$$\begin{aligned} p_{ij}^{(n)} = \mathbb{P}(X_n = j \mid X_0 = i) = \sum_{r=1}^{n} \mathbb{P}(\tau_j = r \mid X_0 = i) p_{jj}^{(n-r)} &\to \pi_j \sum_{r=1}^{\infty} \mathbb{P}(\tau_j = r \mid X_0 = i) \\ &= \pi_j \mathbb{P}(\tau_j < \infty \mid X_0 = i). \end{aligned}$$

Exercise 5. Let X be a Markov chain with transition matrix P on $E = \{1, 2, 3, 4, 5\}$ given by

$$P = \begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} & 0 & \frac{3}{4} \end{pmatrix}.$$

(a) Find the communicating classes of P. For the recurrent classes, find the corresponding stationary distributions.

(b) Supposing that $X_0 \sim \alpha$ for a distribution α on E, find the limiting distribution of X_n when $n \to \infty$.
Hint: Suppose that X starts in a transcient state of E and find the limiting distribution in this case.

Solution. (a) The communicating classes of P are $\{1, 4\}$, $\{3, 5\}$ et $\{2\}$. Closed (and recurrent) classes are $\{1, 4\}$ et $\{3, 5\}$. The submatrix P_1 corresponding to $\{1, 4\}$ is given by:

$$P_1 = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

The stationary distribution of the system $\pi_1 = (a, b)$ verifies

$$\begin{cases} \frac{a}{3} + \frac{b}{2} &= a, \\ \frac{2a}{3} + \frac{b}{2} &= b, \\ a + b &= 1. \end{cases}$$

The solution of this system is given by $\pi_1 = (\frac{3}{7}, \frac{4}{7})$.
The submatrix P_2 corresponding to $\{3, 5\}$ is given by:

$$P_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

The stationary distribution of the system $\pi_2 = (c, d)$ verifies

$$\begin{cases} \frac{c}{2} + \frac{d}{4} & = c, \\ \frac{c}{2} + \frac{3d}{4} & = d, \\ c + d & = 1. \end{cases}$$

The solution of this system is given by $\pi_2 = (\frac{1}{3}, \frac{2}{3})$.

(b) State 2 is the only transient state of the system. Starting from 2, we know that $P_{2i}^n \overset{n\to\infty}{\longrightarrow} h(i)\pi(i)$ where

$$h(i) =: \mathbb{P}(T_i < \infty \mid X_0 = 2),\ \ i = 1, 3, 4, 5,$$

and $\pi(i)$ corresponds to the component of the stationary distribution relative to state i $(\pi(1) = \pi_1(1), \pi(3) = \pi_2(1), \pi(4) = \pi_1(2), \pi(5) = \pi_2(2))$.

We need to compute $h(i)$ for $i = 1, 3, 4, 5$. Since $\{1, 4\}$ is a closed and recurrent class, we have that $\mathbb{P}(T_1 < \infty \mid X_0 = 4) = 1$. Since $\{3, 5\}$ is closed, we also have that $\mathbb{P}(T_1 < \infty \mid X_0 = 3) = 0$. Using this, we obtain:

$$h(1) = \frac{1}{4}h(1) + \frac{1}{2} \cdot 0 + \frac{1}{4} \cdot 1 \Longrightarrow h(1) = \frac{1}{3}.$$

Similarly, we find:

$$\begin{cases} h(3) = \frac{1}{4}h(3) + \frac{1}{2} & \Longrightarrow h(3) = \frac{2}{3}, \\ h(4) = \frac{1}{4}h(4) + \frac{1}{4} & \Longrightarrow h(4) = \frac{1}{3}, \\ h(5) = \frac{1}{4}h(5) + \frac{1}{2} & \Longrightarrow h(5) = \frac{2}{3}. \end{cases}$$

Since 2 is transient, $P_{22}^n \overset{n\to\infty}{\longrightarrow} 0$. We thus get:

$$P_{21}^n \to \frac{1}{3} \times \frac{3}{7} = \frac{1}{7}, P_{23}^n \to \frac{2}{9}, P_{24}^n \to \frac{4}{21}, P_{25}^n \to \frac{4}{9}.$$

Therefore, the transition matrix P^n converges to P_∞ given by:

$$P_\infty = \begin{pmatrix} \frac{3}{7} & 0 & 0 & \frac{4}{7} & 0 \\ \frac{1}{7} & 0 & \frac{2}{9} & \frac{4}{21} & \frac{4}{9} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{3}{7} & 0 & 0 & \frac{4}{7} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}.$$

So, if $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ is the initial distribution of X, the limiting distribution of X_n, denoted by α_∞, is given by

$$\alpha_\infty = \alpha P_\infty = (\frac{3\alpha_1}{7} + \frac{\alpha_2}{7} + \frac{3\alpha_4}{7}, 0, \frac{2\alpha_2}{9} + \frac{\alpha_3}{3} + \frac{\alpha_5}{3}, \frac{4\alpha_1}{7} + \frac{4\alpha_2}{21} + \frac{4\alpha_4}{7}, \frac{4\alpha_2}{9} + \frac{2\alpha_3}{3} + \frac{2\alpha_5}{3}).$$

Exercise 6. Let $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ be two independent Markov chains, aperiodic and irreducible, defined on the state spaces E and E', respectively. Show that $(X_n, Y_n)_{n\geq 0}$ is an aperiodic and irreducible Markov chain on $E \times E'$. Find an example of $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ independent and irreducible, but for which $(X_n, Y_n)_{n\geq 0}$ is not irreducible.

Solution. We write $(p_{ij})_{i,j\in E}$ and $(q_{ij})_{i,j\in E'}$ for the transition probabilities for $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ respectively. For all $i_0, i \in E$, $j_0, j \in E'$, there exist $r > 0$ and $s > 0$ with $p^{(r)}_{i_0 i} > 0$ and $q^{(s)}_{j_0 j} > 0$. If X_n and Y_n are aperiodic, then, by a theorem of the class, there exists n_0 such that for all $n \geq n_0$, we have $p^{(n)}_{ii} > 0$ and $q^{(n)}_{jj} > 0$. Thus, for all $m \geq r + s + n_0$, we have that $p^{(m)}_{i_0 i} > 0$ and that $q^{(m)}_{j_0 j} > 0$, and thus

$$P\Big\{(X_m, Y_m) = (i, j) | (X_0, Y_0) = (i_0, j_0)\Big\} = p^{(m)}_{i_0 i} q^{(m)}_{j_0 j} > 0.$$

Notice that this implies that the periodicity of $(X_n, Y_n)_{n\geq 0}$ is equal to 1.
For the counterexample, we consider $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ to be two independent random walks on $\mathbb{Z}$. Notice that if we start at $(0, 0)$, we can only reach vertices for which the sum of its coordinates is even. Hence (X_n, Y_n) is not irreducible.

Exercise 7. (**Branching process with immigration**) For $n \in \mathbb{N}$, let $(N^n_k)_{k\geq 0}$ be a sequence of independent random variables on $\mathbb{Z}^+$ with a common generating function $\phi(t) = E(t^{N^n_k})$. The branching process with immigration is defined as

$$X_n = N^n_1 + \ldots + N^n_{X_{n-1}} + I_n, \qquad n \geq 0,$$

where $(I_n)_{n\geqslant 0}$ is a sequence of independent random variables with values in $\mathbb{Z}^+$ with a common generating function $\psi(t) = E(t^{I_n})$. Show that if $X_0 = 1$ then

$$E(t^{X_n}) = \phi^{(n)}(t) \prod_{k=0}^{n-1} \psi(\phi^{(k)}(t)).$$

In the case where the number of immigrants in each generation is a Poisson random variable of parameter λ and $P(N^n_k = 0) = 1 - p$, $P(N^n_k = 1) = p$, find the proportion of time in the long run for which the population is 0.

Solution. The equation for $\mathbb{E}(t^{X_n})$ can be proved by induction by doing a one step decomposition. Indeed, for $n = 1$, the result is straightforward by independence

$$\mathbb{E}(t^{X_1}) = \mathbb{E}(t^{N^1 + I_1}) = \mathbb{E}(t^{N^1})\mathbb{E}(t^{I_1}) = \phi(t)\psi(t).$$

Suppose that this holds for n. We obtain

$$
\begin{aligned}
\mathbb{E}(t^{X_{n+1}}) = \mathbb{E}(\mathbb{E}(t^{X_{n+1}} | \, X_n)) = & \\
&= \sum_{k=0}^{\infty} \mathbb{E}(t^{X_{n+1}} | \, X_n = k)\mathbb{P}(X_n = k) = \\
&= \sum_{k=0}^{\infty} \mathbb{E}(t^{N_1^{n+1}+\ldots+N_k^{n+1}+I_{n+1}})\mathbb{P}(X_n = k) = \\
&= \psi(t) \sum_{k=0}^{\infty} \phi(t)^k \mathbb{P}(X_n = k) = \\
&= \psi(t)\mathbb{E}(\phi(t)^{X_n}).
\end{aligned}
$$

We finally get, by using the induction relation:

$$
\mathbb{E}(t^{X_n}) = \phi^{(n)}(t) \prod_{k=0}^{n-1} \psi(\phi^{(k)}(t)).
$$

In the special case of immigration that has a Poisson law, we get:

$$
\mathbb{E}(t^{X_n}) = (1 + p^n(t-1)) \exp\left(\lambda(t-1)\frac{1-p^n}{1-p}\right).
$$

We conclude for $0 \leq p < 1$

$$
\begin{aligned}
\lim_{n\to\infty} \mathbb{P}(X_n = 0) &= \lim_{n\to\infty} \lim_{t\to 0} \mathbb{E}(t^{X_n}) \\
&= \lim_{n\to\infty} (1-p^n) \exp\left(-\lambda \frac{1-p^n}{1-p}\right) \\
&= \exp\left(-\frac{\lambda}{1-p}\right).
\end{aligned}
$$

That is the proportion of time for which the population is 0 since

$$
\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{P}(X_k = 0) = \lim_{n\to\infty} \mathbb{P}(X_n = 0) = \exp\left(-\frac{\lambda}{1-p}\right).
$$

Exercise 8. (**Metropolis–Hastings algorithm**) Suppose that we have a distribution p (called target distribution) on a countable space E. Then, for each $x \in E$, let q_x be a distribution on E (called the proposal distribution) with $q_x(y) > 0$ whenever $q_y(x) > 0$, for all $y \in E$. The Metropolis–Hastings algorithm constructs a Markov chain $(X_n)_{n\geq 0}$ as follows:

(i). Let $X_0 = x_0 \in E$ be random fixed state.

(ii). For $X_n = x$, choose a candidate y according to the proposal distribution q_x. In other words, with probability $q_x(y)$, the state y is the candidate state to which we may jump at time $n+1$. Once we have a candidate state, we will decide if we jump to it or stay at x in the following way: Let U be a uniform random variable on $[0,1]$, the variable X_{n+1} is defined as

$$
X_{n+1} = \begin{cases} y & \text{if } U \leq \min\left(\frac{p(y)q_y(x)}{p(x)q_x(y)}, 1\right) \\ x & \text{otherwise.} \end{cases}
$$

Show that if $(X_n)_{n\geq 0}$ is irreducible and aperiodic, then it is a reversible chain with respect to its stationary distribution p.

Solution. Note that in general, irreducibility and aperiodicity are simple to show given the proposal distribution. In particular, note that if there exist x, y such that the ratio $\frac{p(y)q_y(x)}{p(x)q_x(y)}$ is not always equal to 1 (which is generally the case), then there is a positive probability to stay in state x, and the chain is therefore aperiodic.
It is easy to show that the transition probability from x to y with $x \neq y$ is

$$p_{xy} = q_x(y) \min\left(\frac{p(y)q_y(x)}{p(x)q_x(y)}, 1\right).$$

With this, we have by the detailed balance (suppose that $p(x)q_x(y) > p(y)q_y(x)$ wlog)

$$\pi_x \frac{q_x(y)\frac{p(y)q_y(x)}{p(x)q_x(y)}}{q_y(x)} = \pi_y$$

and thus $\pi_x = p(x)$ for all $x \in \mathbb{E}$. Moreover, by the previous exercise sheet, the chain is reversible with respect to its stationary distribution.