Exercise 1. Consider the following transition matrix:

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Determine which states are recurrent and which are transient.

Solution. First we can find the communicating classes: $\{1,5\}$, $\{3\}$, $\{2,4\}$. If we start in the class $\{1,5\}$, we will remain in this class forever, in other words this class is closed. We obviously have that $\{3\}$ is closed too. We know that all states in a finite closed communicating class are recurrent. We then deduce that the states 1,3 and 5 are recurrent.
On the other hand, starting from 2 or 4, there's a positive probability to go to state 3 and hence never go back to 2 or 4. More precisely, writing T_2 for the number of steps to hit 2, we have

$$\mathbb{P}(T_2 = \infty \mid X_0 = 2) \geq P_{2,4}P_{4,3} > 0.$$

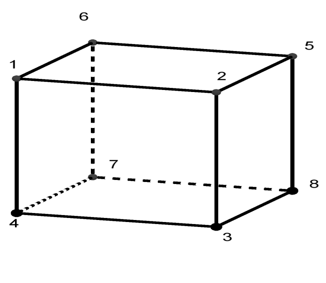
We deduce (using a similar argument for 4) that states 2 and 4 are transient.

Exercise 2. A particle moves on the eight vertices of a cube in the following way: at each step the particle is equally likely to move to each of the three adjacent vertices, independently of its past motion. Let the vertex 1 be the initial vertex occupied by the particle. Calculate each of the following quantities:

(a) the expected number of steps until the particle returns to 1,

(b) the expected number of visits to 8 until the first return to 1,

(c) the expected number of steps until the first visit to 8.



Solution. (a) Let $g(i) := \mathbb{E}[T_1 \mid X_0 = i]$ where T_1 is the first time to hit state 1 for the chain $(X_n)_{n\geq 0}$ that has values on the vertices $\{1, \cdots, 8\}$ of the cube. By symmetry, we see easily that we have $g(2) = g(4) = g(6)$ and $g(3) = g(5) = g(7)$. Using this, we get:

$$\begin{aligned}
g(1) &= 1 + \frac{1}{3}g(2) + \frac{1}{3}g(4) + \frac{1}{3}g(6) = 1 + g(2) \\
g(2) &= 1 + \frac{1}{3}g(3) + \frac{1}{3}g(5) = 1 + \frac{2}{3}g(3) \\
g(3) &= 1 + \frac{1}{3}g(2) + \frac{1}{3}g(4) + \frac{1}{3}g(8) = 1 + \frac{2}{3}g(2) + \frac{1}{3}g(8) \\
g(8) &= 1 + g(3).
\end{aligned}$$

Solving this, we get that $g(1) = 8$.

(b) Let $k(i) = \mathbb{E}[V_8 \mid X_0 = i]$ for $i = 1, \cdots, 8$ where V_8 represents the number of visits to state 8 before returning to 1. Using the symmetry of the cube, we have in this case that

$$\begin{aligned}
k(1) &= \frac{1}{3}k(2) + \frac{1}{3}k(4) + \frac{1}{3}k(6) = k(2) \\
k(2) &= \frac{2}{3}k(3) \\
k(3) &= \frac{2}{3}k(2) + \frac{1}{3}(1 + k(8)) \\
k(8) &= k(3).
\end{aligned}$$

Solving this, we get $k(1) = 1$.

(c) Let T_8 be the first time to hit 8 and $l(i) := \mathbb{E}[T_8 \mid X_0 = i]$ for $i = 1, \cdots, 8$. Then we have

$$\begin{aligned}
l(1) &= 1 + l(2) \\
l(2) &= 1 + \frac{1}{3}l(1) + \frac{2}{3}l(3) \\
l(3) &= 1 + \frac{2}{3}l(2).
\end{aligned}$$

Solving this, we get that $l(1) = 10$.

Exercise 3. (a) A transition matrix P defined on a state space E and a distribution λ have the *detailed balance property* if

$$\lambda_j P_{ji} = \lambda_i P_{ij}, \ \forall i, j \in E.$$

Show that in this case, λ is a stationary distribution for P.

(b) Consider two urns each of which contains m balls; b of these $2m$ balls are black, and the remaining $2m - b$ are white. We say that the system is at state i if the first urn contains i black balls and $m - i$ white balls while the second contains $b - i$ black balls and $m - b + i$ white balls. Each trial consists of choosing a ball at random from each urn and exchanging the two. Let X_n be the state of the system after n exchanges have been made. X_n is a Markov chain.

(1) Compute its transition probability.

(2) Verify (using (a)) that the stationary distribution is given by

$$\pi(i) = \frac{\binom{b}{i}\binom{2m-b}{m-i}}{\binom{2m}{m}}.$$

(3) Can you give a simple intuitive explanation why the formula in (2) gives the right answer?

Solution. (a) We need to show that $\lambda P = \lambda$. Using the detailed balance equations, we get easily for all $i \in I$:

$$(\lambda P)_i = \sum_{j \in I} \lambda_j p_{ji} = \sum_{j \in I} \lambda_i p_{ij} = \lambda_i.$$

(b) (1) Let $p(i, i+1)$ be the probability to go from i black balls to $i+1$ in the first urn after one step (for $i \leq b \wedge m$). This events happens if and only if we choose a white ball in the first urn and a black ball in the second one, so we have

$$p(i, i+1) = \frac{m-i}{m} \cdot \frac{b-i}{m}.$$

Similarly, we have $p(i, i-1) = \frac{i}{m} \cdot \frac{m-b+i}{m}$. After one step, the number of black balls can remain unchange i, or go to either $i+1$ or $i-1$. Therefore

$$p(i,i) = 1 - p(i, i+1) - p(i, i-1) = \frac{i}{m} \cdot \frac{b-i}{m} + \frac{m-i}{m} \cdot \frac{m-b+i}{m}.$$

(2) A sufficient condition for π to be a stationary distribution is to verify the detailed balance property:

$$\pi(i)p(i, i+1) = \pi(i+1)p(i+1, i), i \in [0, b \wedge m]. \quad (1)$$

If $|i-j| > 1$, the equalities $\pi(i)p(i,j) = \pi(j)p(j,i)$ are clearly satisfied since $p(i,j) = p(j,i) = 0$. By developping the left term in (1), we get

$$\begin{aligned}
\binom{2m}{m}\pi(i)m^2 p(i, i+1) &= \binom{b}{i}\binom{2m-b}{m-i}(m-i)(b-i) \\
&= \frac{b!}{i!(b-i-1)!} \cdot \frac{(2m-b)!}{(m-i-1)!(m-b+i)!} \\
&= \binom{b}{i+1}(i+1)\binom{2m-b}{m-i-1}(m-b+i+1) \\
&= m^2 \cdot \binom{b}{i+1}\binom{2m-b}{m-i-1}\frac{i+1}{m}\frac{m-b+i+1}{m} \\
&= \binom{2m}{m}\pi(i+1)m^2 p(i+1, i).
\end{aligned}$$

This shows that π verifies the detailed balance equations and so is the unique stationary distribution of the system.

(3) We know that if X_0 is distributed according to the stationary distribution π, then X_n is also distributed according to π for all $n \geq 1$. Suppose that we number the balls from 1 to $2m$ and we arrange them randomly (by a permutation $\sigma \in S_{2m}$) and we put the first m balls in the first urn (this corresponds to the definition of π in (b)). By exchanging two balls randomly chosen from the first and second urn, the new setting of the balls is "as random as" before. In other words, if the $2m$ balls are arranged in a uniform way at time $t = 0$, they will intuitively still be uniformly ordered after one step of the process.

Exercise 4. Consider a Markov chain with state space $S = \{1, 2\}$ and transition matrix

$$\begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix},$$

$0 < a, b < 1$. Use the Markov property to show that

$$\mathbb{P}(X_{n+1} = 1) - \frac{b}{a+b} = (1-a-b)\{\mathbb{P}(X_n = 1) - \frac{b}{a+b}\},$$

and conclude that

$$\mathbb{P}(X_n = 1) = \frac{b}{a+b} + (1-a-b)^n\{\mathbb{P}(X_0 = 1) - \frac{b}{a+b}\}.$$

Further show that $\mathbb{P}(X_n = 1)$ converges exponentially fast to its limit distribution $b/(a+b)$.

Solution. By the law of total probabilities and using the given transition matrix, we have for all $n \geq 0$,

$$\begin{aligned}
\mathbb{P}(X_{n+1} = 1) &= \mathbb{P}(X_{n+1} = 1 \mid X_n = 1) \cdot \mathbb{P}(X_n = 1) + \mathbb{P}(X_{n+1} = 1 \mid X_n = 2) \cdot \mathbb{P}(X_n = 2) \\
&= (1-a) \cdot \mathbb{P}(X_n = 1) + b \cdot \mathbb{P}(X_n = 2) \\
&= b + (1-a-b) \cdot \mathbb{P}(X_n = 1).
\end{aligned}$$

substracting $\frac{b}{a+b}$ from both side, we obtain

$$\mathbb{P}(X_{n+1} = 1) - \frac{b}{a+b} = -\frac{b}{a+b} + b + (1-a-b) \cdot \mathbb{P}(X_n = 1) = (1-a-b) \cdot [\mathbb{P}(X_n = 1) - \frac{b}{a+b}].$$

Let us use induction for the second equation. It is clearly true for $n = 0$. Assume it is true for n, we have

$$\begin{aligned}
\mathbb{P}(X_{n+1} = 1) &= \frac{b}{a+b} + (1-a-b) \cdot [\mathbb{P}(X_n = 1) - \frac{b}{a+b}] \\
&= \frac{b}{a+b} + (1-a-b) \cdot \Big((1-a-b)^n [\mathbb{P}(X_0 = 1) - \frac{b}{a+b}]\Big) \\
&= \frac{b}{a+b} + (1-a-b)^{n+1} [\mathbb{P}(X_0 = 1) - \frac{b}{a+b}].
\end{aligned}$$

We deduce that the equality is true for all $n \geq 0$, and so $\mathbb{P}(X_n = 1)$ converges towards $\frac{b}{a+b}$ as n grows to infinity in the case where $0 < a + b < 2$.

Exercise 5. (**Reversible Processes**)

a) Let P be an irreducible matrix with stationary distribution π. We assume that $(X_n)_{0\le n\le N}$ is Markov(π, P). The process $Y_n = X_{N-n}$, $0 \le n \le N$ is called the *reverse process* of $(X_n)_{0\le n\le N}$. Show that $(Y_n)_{0\le n\le N}$ is Markov$(\pi, \hat{P})$, where $\hat{P} = (\hat{p}_{ij})$ is given by

$$\pi_j \hat{p}_{ji} = \pi_i p_{ij}, \ \forall\ i, j,$$

and $\hat{P}$ is also irreducible with stationary distribution π.

b) A transition matrix P is said to be *doubly stochastic* if its columns sum also to 1, that is $\sum_i p_{ij} = 1$ for all j.
Show that the stationary distribution of an irreducible Markov chain on N states is the uniform distribution ($\pi(i) = \frac{1}{N},\ 1 \le i \le N$) if and only if its transition matrix is doubly stochastic.

c) We say that an irreducible Markov chain $X \sim$ Markov(λ, P) is *reversible* if $\hat{P} = P$ (in that case λ should be stationary). Find an irreducible chain on $E = \{1, 2, 3\}$ with a stationary distribution but not reversible.

Solution. a) We first verify that $\hat{P}$ is indeed a stochastic matrix:

$$\sum_{i\in I} \hat{p}_{ji} = \frac{1}{\pi_j} \sum_{i\in I} \pi_i p_{ij} = 1,$$

where I is the set of states, and where we used that π is an invariant distribution under P. We verify now that π is also a stationary distribution of $\hat{P}$:

$$\sum_{j\in I} \pi_j \hat{p}_{ji} = \sum_{j\in I} \pi_i p_{ij} = \pi_i.$$

We have by the Markov property

$$\begin{aligned}\mathbb{P}(Y_0 = i_0, Y_1 = i_1, \cdots, Y_N = i_N) &= \mathbb{P}(X_0 = i_N, X_1 = i_{N-1}, \cdots, X_N = i_0)\\ &= \pi_{i_N} p_{i_N i_{N-1}} \cdots p_{i_1 i_0} = \pi_{i_0} \hat{p}_{i_0 i_1} \cdots \hat{p}_{i_{N-1} i_N}.\end{aligned}$$

This shows that $(Y_n)_{0\le n\le N}$ is $Markov(\pi, \hat{P})$.
Finally, since P is irreducible, for all pair of states $i, j \in I$, there exists a pair of states $i_0 = i, i_1, \cdots i_n = j$ with $p_{i_0 i_1} \cdots p_{i_{n-1} i_n} > 0$. We finally get:

$$\hat{p}_{i_n i_{n-1}} \cdots \hat{p}_{i_1 i_0} = \pi_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} i_n} / \pi_{i_n} > 0,$$

we deduce that $\hat{P}$ is irreducible.

b) Suppose first that $\pi = (\frac{1}{N}, \cdots, \frac{1}{N})$ is the stationary distribution corresponding to P. Then we have, for $1 \le i, j \le N$

$$\frac{1}{N} = \pi(j) = \sum_i \pi(i) p_{ij} = \frac{1}{N} \sum_i p_{ij} \quad \Longrightarrow \sum_i p_{ij} = 1.$$

Conversely, suppose that P is doubly stochastic. To show that the uniform distribution $\lambda(i) = \frac{1}{N}$ is the stationary distribution in this case, we verify that λ verifies

$$\sum_i \lambda(i) p_{ij} = \frac{1}{N} \times \sum_i p_{ij} = \frac{1}{N} = \lambda(j),\ 1 \le j \le N.$$

Since the stationary distribution π is unique, we deduce that $\pi \equiv \lambda$, and so the uniform distribution is indeed the stationary distribution of the system in this case.

c) Consider the Markov chain on $E = \{1, 2, 3\}$ with transition matrix P given by

$$P = \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}.$$

By part b), we have that $\pi = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is the stationary distribution corresponding to P. We obtain in this case that $\hat{P} = P^T$. As P is not symmetric, $\hat{P} \neq P$ and so the chain is not reversible.

Exercise 6. Consider two boxes filled with gas molecules and joined by a small gap allowing them to pass from one box to the other. Assume that in total N molecules are in this configuration. We model the system so that at each time only one (randomly chosen) molecule is able to move from one box to the other.

(1) Show that the number of molecules in a box evolves according to a Markov process.

(2) Give the transition probabilities.

(3) What is the stationary distribution (detailed balance equations)?

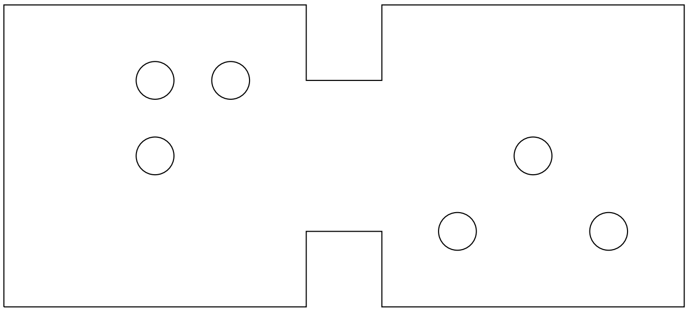


Figure 1: Configuration of the problem.

Solution. (1-2) One update of the system consists in a transition of a randomly selected particle moving from one box to the other. If we denote by X_n the number of molecules in the box A, the possible states are $S = \{0, 1, \dots N\}$. The transitions of the system are given by $p_{r,r+1} = 1 - \frac{r}{N}$ and $p_{r,r-1} = \frac{r}{N}$.

(3) We can guess that the equilibrium should be $\pi_r = 2^{-N}\binom{N}{r}$, it is then sufficient to verify that $\forall r$

$$\pi_r = \sum p_{vr}\pi_v = \pi_{r-1}p_{r-1,r} + \pi_{r+1}p_{r+1,r}.$$

Otherwise we can start from detailed balance equations $0 \leqslant r \leqslant N$:

$$\begin{aligned} \pi_{r-1}p_{r-1,r} &= \pi_r p_{r,r-1} \\ \Leftrightarrow \pi_{r-1}\frac{N-r+1}{N} &= \pi_r \frac{r}{N} \\ \Rightarrow \pi_r &= \frac{N-r+1}{r}\pi_{r-1} \\ \Rightarrow \pi_r &= \frac{N!}{r!(N-r)!}\pi_0. \end{aligned}$$

Then using that π should be a probability, we find $\pi_0 = \frac{1}{2^N}$. We finally get that:

$$\pi_r = \frac{1}{2^N}\binom{N}{r}.$$

Exercise 7. Consider the aging chain on $\{0, 1, 2, \cdots\}$ in which for any $n \geq 0$ the individual gets one day older from n to $n+1$ with probability p_n but dies and returns to age 0 with probability $1 - p_n$. Find conditions that guarantee that

(a) 0 is recurrent,

(b) 0 is positive recurrent.

(c) Find the stationary distribution of the chain.

Solution. (a) Let $T_0 = \min\{n \geq 1 \mid X_n = 0\}$. By definition, the state 0 is recurrent if and only if $\mathbb{P}_0(T_0 < \infty) = 1$. We have in this case

$$\mathbb{P}_0(T_0 < \infty) = 1 - \mathbb{P}_0(T_0 = \infty) = 1 - \prod_{i=0}^{\infty} p_i.$$

Therefore, 0 is recurrent if and only if $\prod_{i=0}^{\infty} p_i = 0$.

(b) By definition, 0 is positive recurrent if and only if $\mathbb{E}_0[T_0] < \infty$. Computing this expectation, we get

$$\begin{aligned}
\mathbb{E}_0[T_0] &= 1 \cdot (1 - p_0) + 2p_0(1 - p_1) + 3p_0p_1(1 - p_2) + \cdots \\
&= \sum_{k=0}^{\infty} (k+1)p_0 \cdots p_{k-1} \cdot (1 - p_k) \\
&= 1 - p_0 + 2p_0 - 2p_0p_1 + 3p_0p_1 - 3p_0p_1p_2 + \cdots \\
&= 1 + p_0 + p_0p_1 + p_0p_1p_2 + \cdots
\end{aligned}$$

Hence, $\mathbb{E}_0[T_0] < \infty$ if and only if $\sum_{k=0}^{\infty} \prod_{i=0}^{k} p_i < \infty$.

(c) We suppose that the chain is positive recurrent. It is straightforward to see that it is irreducible. Then it has a unique stationary distribution π that satisfies, for all $x \geq 0$,

$$\begin{cases} \sum_{y \geq 0} p(y, x)\pi(y) & = \pi(x), \\ \sum_{y \geq 0} \pi(y) & = 1. \end{cases}$$

We get $\pi(k+1) = p_k \pi(k)$ for all $k \geq 0$, and so, by writing $\pi(0) = c$, we get $\pi(k+1) = \prod_{i=0}^{k} p_i c$. Since the sum over all the components of π is 1, we get $c = (1 + p_0 + p_0p_1 + \cdots)^{-1}$. Notice that $\pi(0) > 0$ by (b).

Exercise 8. (**Random walk on a graph**)
An undirected graph $\mathcal{G}$ is a countable collection of states (that we call vertices) along with some edges connecting them. The degree d_i of a vertex i is the number of edges incident to i. We suppose the graph to be locally finite (i.e., each edge is incident to a finite number of

edges). We say that a Markov chain on the state space $E = \mathcal{G}$ is a random walk on the graph if the transition probabilities are given by

$$p_{i,j} = \begin{cases} 1/d_i & \text{if } (i,j) \text{ is an edge,} \\ 0 & \text{otherwise,} \end{cases}$$

for $i, j \in \mathcal{G}$.

a) We assume that $\mathcal{G}$ is connected (implying that P is irreducible) and that $\sum_i d_i < \infty$. Find the stationary distribution of the random walk on $\mathcal{G}$.
Hint: Assume that the random walk is reversible and find a stationary distribution verifying the detailed balance equations. Explain why P is reversible.

b) We assume now that the graph is a chessboard, i.e., the vertices are $\mathcal{G} = \{1, \ldots, 8\}^2$ and the edges are the possible moves of a King. We assume that the King starts its random walk in one of the four corners of the chessboard $c \in \mathcal{G}$. Compute the mean return time to the initial state $\mathbb{E}_c(T_c)$ of the King. Compute the same quantity for a Knight instead of a King.

Solution. a) We start from the detailed balance equations:

$$\pi_i P_{ij} = \pi_j P_{ji}, \ \forall i, j \in \mathcal{G}. \tag{2}$$

This is equivalent to

$$\pi_i \cdot \frac{\mathbb{1}_{\{i \sim j\}}}{d_i} = \pi_j \cdot \frac{\mathbb{1}_{\{j \sim i\}}}{d_j},$$

where $\{i \sim j\}$ indicate the event where i and j are connected by an edge. If $i \sim j$, we have

$$\frac{\pi_i}{d_i} = \frac{\pi_j}{d_j}. \tag{3}$$

Otherwise, since the graph is connected, there exists a path $k_0 = i \sim k_1 \sim \cdots \sim k_m = j$ joining i and j. For all couple of points k_r, k_{r+1} the equality (3) is verified (for π_{k_r} et $\pi_{k_{r+1}}$). We deduce then that (3) is verified for all i and j of the graph. Using that $\sum_{i \in \mathcal{G}} \pi_i = 1$, we get easily that $\pi_i = \frac{d_i}{\sum_{j \in \mathcal{G}} d_j}$.

b) By a theorem seen in the course, we know that $\mathbb{E}_c[T_c] = \frac{1}{\pi_c} = \frac{\sum_{j \in \mathcal{G}} d_j}{d_c}$. It remains to find the degrees of all the vertices of the graph. If we are in one of the four corners, the degree (= the number of possible moves of the King) is equal to 3. If we are on one of the 24 boxes at the border of the chessboard and which is not one of the 4 corners, the degree of one of the box is 5. In all remaining states (for the 64-4-24=36 boxes remaining), the degree is 8. We get then:

$$\mathbb{E}_c[T_c] = \frac{\sum_{j \in \mathcal{G}} d_j}{d_c} = \frac{4 \cdot 3 + 24 \cdot 5 + 8 \cdot 36}{3} = 140.$$

Starting from a corner c, we need in average 140 moves of the King to come back to c.
The random walk of the Knight: counting the number of possible moves of the Knight starting from one of the 64 vertices of the graph, one gets the following configuration for the degrees:

2	3	4	4	4	4	3	2
3	4	6	6	6	6	4	3
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
3	4	6	6	6	6	4	3
2	3	4	4	4	4	3	2

Starting from one of the corners c, the expected time before return to c is given by:

$$\mathbb{E}_c[T_c] = \frac{1}{\pi_c} = \frac{\sum_{j \in \mathcal{G}} d_j}{d_c} = \frac{2 \cdot 4 + 3 \cdot 8 + 4 \cdot 20 + 16 \cdot 6 + 16 \cdot 8}{2} = 168.$$