Exercise 1 (Random walk). Let $(X_n)_{n\geq 0}$ be a one-dimensional random walk on the state space $\mathbb{Z}$ defined by the following transition probabilities:

$$P_{xy} = \begin{cases} p, & y = x+1, \\ q, & y = x-1, \end{cases}$$

(1) Prove that the random walk is recurrent if an only if $p = q$.

Hint: Note that $p_{00}^{2n+1} = 0$ for all $n \in \mathbb{N}$, and find the probability p_{00}^{2n}. You can then use Stirling's approximation to $n!$

$$n! \sim \sqrt{2\pi n}(n/e)^n, \quad n \to \infty.$$

(2) In the transient case $p \neq q$, find the limit $\lim_{n\to\infty} X_n$.

Solution. (1) This Markov chain is irreducible. Suppose we start at 0, then $p_{00}^{(2n+1)} = 0$ for all n. Any given sequence of $2n$ steps from 0 to 0 has probability $p^n q^n$ and the number of sequences is the number of ways of choosing n steps up from $2n$ steps is. Thus

$$p_{00}^{(2n)} = \binom{2n}{n} p^n q^n.$$

We will use Stirling's approximation to $n!$

$$n! \sim \sqrt{2\pi n}(n/e)^n, \quad n \to \infty.$$

With this we obtain

$$p_{00}^{(2n)} = \frac{(2n)!}{(n!)^2}(pq)^n \sim \frac{C(4pq)^n}{\sqrt{n}}.$$

In the symmetric case $p = q = 1/2$, $4pq = 1$ and so

$$\sum_{n=0}^{\infty} p_{00}^{(2n)} \approx C \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}} = \infty,$$

showing that the random walk is recurrent.

If $p \neq q$, then $4pq = r < 1$ and thus

$$\sum_{n=0}^{\infty} p_{00}^{(2n)} \approx C \sum_{n=0}^{\infty} \frac{r^n}{\sqrt{n}} < \infty,$$

where $C > 0$ is a constant. Thus the random walk is transient.

(2) The strong law of large numbers gives us that depending on the sign of $p - q$

$$\lim_{n\to\infty} X_n \overset{\text{a.s.}}{=} \operatorname{sgn}(p-q)\infty,$$

(please refer to the solution of the next exercice).

Exercise 2 (**Birth and Death chain**). Let us consider a Markov chain $(X_n)_{n\geq 0}$ on the state space $\mathbb{N}$ defined by the following transition probabilities:

$$p(x,y) = \begin{cases} p & \text{if } x>0,\ y=x+1, \\ q & \text{if } x>0,\ y=x-1, \\ 1 & \text{if } x=0,\ y=1. \end{cases}$$

Prove that:

(1) when $p \leq q$ the chain is recurrent.

Hint: study the probability $u(k) = P_k\left(X_n \neq 0, \forall n \in \mathbb{N}\right)$ by showing that

$$u(k+1) - u(k) = \frac{q}{p}\left(u(k) - u(k-1)\right).$$

(2) when $q < p$ the chain is transient.

Hint: consider writing the chain as $X_n = \sum_{i=1}^n Y_i \mathbb{1}(X_{i-1} > 0) + |Y_i| \mathbb{1}(X_{i-1} = 0)$ where

$$Y_i \overset{\text{i.i.d.}}{\sim} \begin{cases} +1 \text{ with prob } p \\ -1 \text{ with prob } q \end{cases},$$

and compare with the biased random walk $\sum_{i=1}^n Y_i$.

Solution. (1) in the case $p \leq q$, we define the probability $u(k) = \mathbb{P}_k\left(X_n \neq 0,\ \forall n \in \mathbb{N}\right)$. The chain is recurrent if and only if $u(k) = 0$ for all $k \in \mathbb{N}$. Clearly $u(0) = 0$, and moreover by the Markov property

$$u(k) = qu(k-1) + pu(k+1),$$

which gives after rearranging

$$u(k+1) - u(k) = \frac{q}{p}(u(k) - u(k-1)) = \left(\frac{q}{p}\right)^k (u(1) - u(0)) = \left(\frac{q}{p}\right)^k u(1).$$

Consequently,

$$u(k+1) = (u(k+1) - u(k)) + (u(k) - u(k-1)) + \cdots + (u(1) - u(0)) = u(1) \sum_{j=0}^{k} \left(\frac{q}{p}\right)^j.$$

Thus, if the sum diverges, i.e., $q \geq p$, then $u(1) = 0 = u(k)$, for all $k \in \mathbb{N}$, since the $u(k)$ must be probabilities.

(2) in the case $q < p$, we will use a coupling argument. Let us consider a sequence of independent and identically distributed random variables $(Y_n)_{n\geq 1}$ such that

$$Y_i \overset{\text{i.i.d.}}{\sim} \begin{cases} +1 \text{ with prob } p \\ -1 \text{ with prob } q \end{cases},$$

This sequence will serve as a common source of randomness to couple the random walk on $\mathbb{Z}$ with the birth and death chain on $\mathbb{N}$. Indeed, if we consider the two processes:

$$X_n = \sum_{i=1}^{n} Y_i \mathbb{1}(X_{i-1} > 0) + |Y_i| \mathbb{1}(X_{i-1} = 0),$$
$$Z_n = \sum_{i=1}^{n} Y_i,$$

we remark that they both evolve according to the common sequence $(Y_n)_{n \geq 1}$ and we can check that X_n is exactly the birth and death chain on $\mathbb{N}$ and Z_n the biased random walk on $\mathbb{Z}$. We can now consider their asymptotic behaviour together. We have by construction the pathwise inequality

$$Z_n(\omega) \leq X_n(\omega), \text{ for all } n \text{ and } \omega \in E.$$

Then, since the expectation of the Y_i's is $p - q > 0$, we can deduce by the law of large numbers that

$$\frac{1}{n} Z_n = \frac{1}{n} \sum_{i=1}^{n} Y_i \overset{\text{a.s.}}{\to} p - q > 0.$$

implying that $Z_n \overset{\text{a.s.}}{\to} +\infty$ and finally $X_n \overset{\text{a.s.}}{\to} +\infty$. We thus conclude that the birth and death chain is transient in this setting.

Exercise 3. Let $Y_1, Y_2, \cdots$ be independent identically distributed random variables with $\mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = -1) = \frac{1}{2}$ and set $X_0 = 1$ and $X_n = X_0 + Y_1 + \cdots + Y_n$ for $n \geq 1$. Define the stopping time

$$H_0 = \inf\{n \geq 0 \mid X_n = 0\}.$$

(a) Find the probability generating function $\phi(s) = \mathbb{E}[s^{H_0}]$.

(b) Suppose that the distribution of the Y_i's is changed to $\mathbb{P}(Y_1 = 2) = \mathbb{P}(Y_1 = -1) = \frac{1}{2}$. Show that ϕ now satisfies

$$s\phi(s)^3 - 2\phi(s) + s = 0.$$

Solution. (a) Suppose that we start at 2 (so $X_0 = 2$ instead of 1) and write H_1 for the first time that $X_n = 1$. By the strong Markov property and conditioning on $H_1 < \infty$ and under $\mathbb{P}_2$, we have that $H_0 = H_1 + \tilde{H_0}$ where $\tilde{H_0}$ is the time needed to reach 0 once we have reached 1 (at time H_1), and $\tilde{H_0}$ is independent of H_1 and is distributed as H_1. So we get

$$\begin{aligned}\mathbb{E}_2[s^{H_0}] &= \mathbb{E}_2[s^{H_1} \mid H_1 < \infty]\, \mathbb{E}_2[s^{\tilde{H_0}} \mid H_1 < \infty] \mathbb{P}_2(H_1 < \infty) \\ &= \mathbb{E}_2[s^{H_1} \mathbb{1}_{H_1 < \infty}]\, \mathbb{E}_2[s^{\tilde{H_0}} \mid H_1 < \infty] \\ &= \mathbb{E}_2[s^{H_1}]^2 = \phi(s)^2.\end{aligned}$$

Returning to our original process $X_n = 1 + Y_1 + \cdots Y_n$, and conditioning on $X_1 = 2$, we have that $H_0 = 1 + \bar{H_0}$ where $\bar{H_0}$ is the time taken, starting from 2 to reach 0, and so is distributed as H_0 under $\mathbb{P}_2$. So we get

$$\begin{aligned}\phi(s) = \mathbb{E}_1[s^{H_0}] &= \frac{1}{2}\,\mathbb{E}_1[s^{H_0} \mid X_1 = 2] + \frac{1}{2}\,\mathbb{E}_1[s^{H_0} \mid X_1 = 0] \\ &= \frac{1}{2}\,\mathbb{E}_1[s^{1+\bar{H}_0} \mid X_1 = 2] + \frac{1}{2}\,\mathbb{E}_1[s^{H_0} \mid X_1 = 0] \\ &= \frac{1}{2}s\,\mathbb{E}_2[s^{H_0}] + \frac{1}{2}s \\ &= \frac{1}{2}s\phi(s)^2 + \frac{1}{2}s.\end{aligned}$$

Solving this, we get

$$\phi(s) = \frac{1 \pm \sqrt{1-s^2}}{s}.$$

Since $\phi(0) \leq 1$ and ϕ is continuous, we get that $\phi(s) = \frac{1-\sqrt{1-s^2}}{s}$ for $s \in]0,1[$.

(b) We use a similar argument as before by applying twice the strong Markov property to get:

$$\mathbb{E}_3[s^{H_0}] = \phi(s)^3.$$

On the other hand, conditioning on $X_1 = 3$ (which happens with probability $\frac{1}{2}$), $H_0 = 1 + H_0'$ where H_0' is the time needed strating from 3 to reach 0 and is distributed as H_0 under $\mathbb{P}_3$. So we get

$$\begin{aligned}\phi(s) = \mathbb{E}_1[s^{H_0}] &= \frac{1}{2}\,\mathbb{E}_1[s^{H_0} \mid X_1 = 3] + \frac{1}{2}\,\mathbb{E}_1[s^{H_0} \mid X_1 = 0] \\ &= \frac{1}{2}\,\mathbb{E}_1[s^{1+H_0'} \mid X_1 = 3] + \frac{1}{2}\,\mathbb{E}_1[s^{H_0} \mid X_1 = 0] \\ &= \frac{1}{2}s\,\mathbb{E}_3[s^{H_0}] + \frac{1}{2}s \\ &= \frac{1}{2}s\phi(s)^3 + \frac{1}{2}s.\end{aligned}$$

Exercise 4. (**Gambler's ruin**) Assume that a gambler is making bets for 1 dollar on fair coin flips, and that she will abandon the game when her fortune falls to 0 or reaches n dollar. Let X_t be the Markov chain on $\{0, \dots, n\}$ describing the gambler's fortune at time t, that is, $\mathbb{P}(X_{t+1} = k+1 \mid X_t = k) = \mathbb{P}(X_{t+1} = k-1 \mid X_t = k) = 1/2$, $k = 1, \dots, n-1$, and $\mathbb{P}(X_{t+1} = 0 \mid X_t = 0) = \mathbb{P}(X_{t+1} = n \mid X_t = n) = 1$. Let T be the time required to be absorbed at one of 0 or n. Assume that $X_0 = k$, where $0 \leq k \leq n$.

(i). Find the probability $\mathbb{P}_k(X_T = n)$ for the gambler to reach n dollars with initial capital k.

(ii). Compute $\mathbb{E}_k[T]$, the expected time to reach n or 0 starting from k.

Solution. (i). Let p_k be the probability that the gambler reaches a fortune of n before ruin, given that she starts with k dollars. We solve simultaneously for $p_0, p_1, \cdots, p_n$. Clearly $p_0 = 0$ and $p_n = 1$. By the total probability formula, we have

$$p_k = \frac{1}{2}p_{k-1} + \frac{1}{2}p_{k+1}, \quad 1 \leq k \leq n-1. \tag{1}$$

It's easy to solve this system to obtain $p_k = \frac{k}{n}$ for $0 \leq k \leq n$ (by defining $\Delta_k := p_k - p_{k-1}$ similarly to the next part). Another method to solve this recursive equation is by using its characteristic equation:

$$x = \frac{1}{2} + \frac{1}{2}x^2 \iff x^2 - 2x + 1 = 0. \tag{2}$$

If we have, in general, two distinct roots $x_1,\ x_2$ for this equation, then the solution of the recursive equation would be of the form

$$p_k = Ax_1^k + Bx_2^k,$$

where A and B are constants.
But in our case, the equation (2) has one root $x_0 = 1$ with multiplicity 2. In this case, the solution for (1) is given by

$$p_k = x_0^k(Ak + B) = Ak + B,$$

where A and B are constants. Since p_0=0 and $p_n = 1$, we get:

$$p_k = \frac{k}{n},\ 0 \leq k \leq n.$$

(ii). We write f_k for the expected time $\mathbb{E}_k[T]$ to be absorbed, starting at position k. Clearly, $f_0 = f_n = 0$. For $1 \leq k \leq n-1$, we have, conditioning on the first step

$$f_k = \frac{1}{2}(1 + f_{k+1}) + \frac{1}{2}(1 + f_{k-1}).$$

To solve this system, we let $\Delta_k = f_k - f_{k-1}$. It is easy to verify that $\Delta_k = \Delta_{k+1} + 2$. Using that $\sum_{k=1}^{n} \Delta_k = 0$, we obtain

$$\Delta_k = n - 1 - 2(k-1),\ \Longrightarrow f_k = k(n-k),\ 0 \leq k \leq n.$$