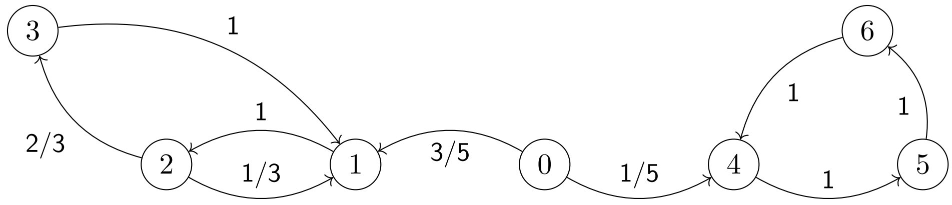
**Exercise 1.** Let $(X_n)_{n\geq 0}$ be a Markov chain determined by the following diagram: Show




that:

(1) Starting from 0, the probability of hitting 6 is $\frac{1}{4}$.

(2) Starting from 1, the probability of hitting 3 is 1.

(3) Starting from 1, it takes on average three steps to hit 3.

**Solution.** (1) According to the diagram, the probability starting from 0 to hit 6 is the same as the probability the we hit 4 (since if we hit 1, we will never hit 6, and if we hit 4, we are sure to hit 6). So this probability is given by

$$\mathbb{P}(0 \to 4) + \mathbb{P}(0 \to 0 \to 4) + \mathbb{P}(0 \to 0 \to 0 \to 4) + \cdots = \frac{1}{5}(1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots) = \frac{1}{4}.$$

(2) Let $h(i) = \mathbb{P}(\text{hitting } 3 \mid X_0 = i)$ for $i = 1, 2, 3$. Then we have

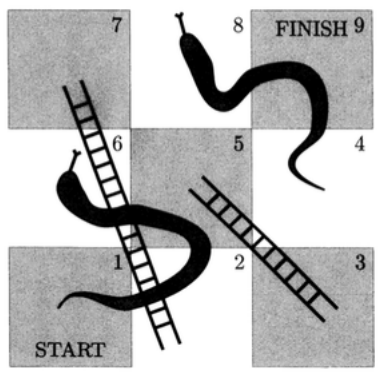
$$h(1) = h(2), \quad h(2) = \frac{1}{3}h(1) + \frac{2}{3} \times 1.$$

So we get that $h(1) = h(2) = 1$.

(3) Let $T$ be the first time that the chain hits state 3 and we write $g(i) = \mathbb{E}[T \mid X_0 = i]$ for $i = 1, 2, 3$. We have that $g(3) = 0$ and so we get

$$g(1) = 1 + g(2), \; g(2) = 1 + \frac{1}{3}g(1) \quad \Longrightarrow g(1) = 3.$$

**Exercise 2.** A simple game of "snakes and ladders" is played on a board of nine squares. At each turn, a player tosses a fair coin and advances one or two places according to whether the coin lands heads or tails. If you land at the foot of a ladder, you climb to the top, but if you land at the head of a snake you slide down to the tail.



(1) How many turns on average does it take to complete the game?

(2) What's the probability that a player who has reached the middle square will complete the game without slipping back to square 1?

**Solution.** (1) Let $X_n$ be our position on the board after $n$ moves and let $g(i) = \mathbb{E}[T \mid X_0 = i]$ where $T$ is the first time that we hit state 9. According to the board and the rules of the game, we have the following equalities:

$$g(3) = g(5),\ g(1) = g(6),\ g(2) = g(7),\ g(4) = g(8),\ g(9) = 0.$$

Using this, we get the following system of equations:

$$\begin{aligned} g(1) &= 1 + \frac{1}{2}g(2) + \frac{1}{2}g(3), \\ g(2) &= g(7) = 1 + \frac{1}{2}g(4), \\ g(3) &= g(5) = 1 + \frac{1}{2}g(1) + \frac{1}{2}g(2), \\ g(4) &= 1 + \frac{1}{2}g(3) + \frac{1}{2}g(1). \end{aligned}$$

From this, we get that $g(1) = 7$.

(2) Let $h(i) = \mathbb{P}(\text{hitting 9 before 1} \mid X_0 = i)$ for $i \le 9$. Then we have $h(9) = 1, h(1) = 0$, $h(8) = h(4),\ h(6) = h(1) = 0$ and

$$h(5) = \frac{1}{2}h(7),\ h(7) = \frac{1}{2}h(4) + \frac{1}{2},\ \ h(4) = \frac{1}{2}h(5).$$

Solving this system of equations, we get that $h(5) = \frac{2}{7}$.

**Exercise 3.** Let $(X_i)_{i\ge 0}$ be a Bernoulli process, which means that the $X_i$'s are i.i.d. with a Bernoulli law of parameter $p$.

(a) Consider the process $(N_n)_{n\ge 0}$ of the number of successes: $N_n$ is the number of successes of the Bernoulli process until time $n$ included.
Show that this process is a Markov chain, compute its transition matrix, draw its corresponding graph and classify the states.

(b) Consider the process $(T_n)_{n\ge 0}$ of the moment of successes: $T_n$ is the time when the $n$th success happens in the Bernoulli process.
Show that this process is a Markov chain, compute its transition matrix, draw its corresponding graph and classify the states.

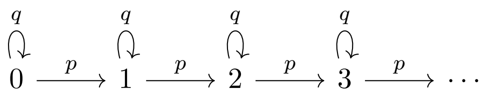
**Solution.** (a) For any integer $n$, we can write $N_{n+1} = N_n + X$ where $X \sim \text{Bernoulli}(p)$ is independent of $N_1, \cdots, N_n$. Thus, knowing $N_n = k$ for some integer $k$, $N_{n+1} = k + X$ is independent of $N_1, \cdots, N_{n-1}$. This shows that $(N_n)_{n\ge 0}$ is a Markov chain. For the homogeneity, it is easy to verify that, for all $n \in \mathbb{N}$,

$$\mathbb{P}(N_{n+1} = j \mid N_n = i) = \begin{cases} p & \text{if } j = i+1, \\ q & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

The transition matrix corresponding to states $\{0, 1, 2, ....\}$ is then given by

$$P = \begin{pmatrix} q & p & 0 & 0 & 0 & \cdots \\ 0 & q & p & 0 & 0 & \cdots \\ 0 & 0 & q & p & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

The associated graph of this Markov chain is given by




All states are transient. Indeed, if we go from $i$ to $i + 1$, we are sure that we are not returning to $i$. Thus, the probability, starting from $i$, to never return to $i$ is strictly positive.

(b) Since the $X_i$'s are i.i.d, we have for all integers $n$

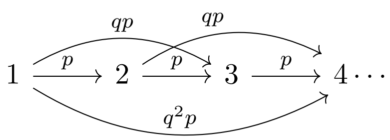
$$T_{n+1} = T_n + S,$$

where S is the first time of a sucess of the Bernoulli process (S is independent of all the $T_i,\ i \leq n$). Hence, conditioning on $T_n$, $T_{n+1}$ is independent of the $T_i$'s for $i \leq n - 1$. It is easy to verify the homogeneity of the Markov chain:

$$\mathbb{P}(T_{n+1} = j \mid T_n = i) = \begin{cases} 0 & \text{if } j \leq i \\ q^{j-i-1}p & \text{otherwise.} \end{cases}$$

The associated transition matrix is then given by

$$Q = \begin{pmatrix} 0 & p & qp & q^2p & q^3p & \cdots \\ 0 & 0 & p & qp & q^2p & \cdots \\ 0 & 0 & 0 & p & qp & \cdots \\ 0 & 0 & 0 & 0 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$



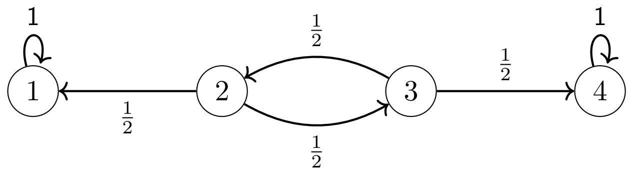

Similarly to the argument of the first part, it's easy to see that all states are transient.

**Exercise 4.** Let $(X_n)_{n\geq 0}$ be a Markov chain determined by the following diagram:
Compute for all $i = 1, 2, 3, 4$ the absorption probability

$$h_i = \mathbb{P}_i\{\exists n \geq 0 : X_n = 4\},$$

i.e. the probability that the chain is absorbed in state 4 knowing that the chain starts at $X_0 = i$. Then compute the mean absorption time knowing that the chain starts in state $i$

$$k_i = \mathbb{E}_i[\inf(n \geq 0 : X_n \in \{1, 4\})].$$




*Proof.* Note that the states 1 and 4 are absorbing. Clearly $h_4 = 1$, moreover as 1 is absorbing, we get $h_1 = 0$. Suppose that the chain is starting at 2 and consider the chain after a transition. The process jumps to 1 with probability 1/2 and to 3 with probability 1/2, then

$$h_2 = \frac{1}{2}h_1 + \frac{1}{2}h_3.$$

By a similar argument, we obtain starting from 3

$$h_3 = \frac{1}{2}h_2 + \frac{1}{2}h_4.$$

The problem is equivalent to solving the following system of equations

$$\begin{cases} h_1 & = 0, \\ h_2 & = 0.5h_1 + 0.5h_3, \\ h_3 & = 0.5h_2 + 0.5h_4, \\ h_4 & = 1. \end{cases}$$

implying $h_2 = 1/3$ and $h_3 = 2/3$.
Let us compute now the mean times spent before absorption. Clearly, $k_1 = 0$ et $k_4 = 0$. By a similar argument as before, we have the equations

$$k_2 = 1 + \frac{1}{2}k_1 + \frac{1}{2}k_3, \;\; k_3 = 1 + \frac{1}{2}k_2 + \frac{1}{2}k_4,$$

where the term 1 is here since we count the first jump. We finally get $k_2 = 2$ and $k_3 = 2$. □

**Exercise 5.** Let $X_0$ be a random variable having values in a countable set $I$. Let $Y_1, Y_2, \ldots$ be a sequence of independent variables, uniformly distributed on $[0, 1]$. Considering any function

$$G : I \times [0, 1] \to I,$$

we define inductively

$$X_{n+1} = G(X_n, Y_{n+1}).$$

(1) Show that $(X_n)_{n \geqslant 0}$ is a Markov chain and write its transition matrix $P$ as a function of $G$.

(2) Can all Markov chains be defined this way?

(3) How do you simulate a Markov chain on a computer?

**Solution.** (1) Writing $\overline{X} = (X_0, \ldots, X_n)$, we have

$$\begin{aligned} & \mathbb{P}\left(X_{n+1} = j | X_n = i, \overline{X}\right) = \\ = \ & \mathbb{P}\left((X_n, Y_{n+1}) \in G^{-1}(\{j\}) | X_n = i, \overline{X}\right) = \\ = \ & \mathbb{P}\left((i, Y_{n+1}) \in \{i\} \times [0,1] \cap G^{-1}(\{j\}) | X_n, \overline{X}\right) = \\ = \ & \mathbb{P}\left(Y_{n+1} \in \pi\left(\{i\} \times [0,1] \cap G^{-1}(\{j\})\right)\right), \end{aligned}$$

by the independence, where $\pi$ is the projection operator defined by $\pi(x,y) = y$ from $I \times [0,1]$ to $[0,1]$.

(2) Yes. Given $(p_{i,j})_{i,j \in I}$, we choose an order (random) $j_1, j_2, \ldots$ of the elements of $I$ (makes sense, since $I$ is countable) and we define:

$$G(i,t) = \begin{cases} j_1, & \text{if } \ 0 \leqslant t \leqslant p_{i,j_1}, \\ j_2, & \text{if } p_{i,j_1} \leqslant t \leqslant p_{i,j_1} + p_{i,j_2}, \\ \cdots \\ j_r, & \text{if } \sum_{n=1}^{r-1} p_{i,j_n} \leqslant t \leqslant \sum_{n=1}^{r} p_{i,j_n}, \\ \cdots \end{cases}$$

Hence

$$\mathbb{P}(X_{n+1} = j_r | X_n = i) = \sum_{n=1}^{r} p_{i,j_n} - \sum_{n=1}^{r-1} p_{i,j_n} = p_{i,j_r},$$

since $Y_1, Y_2, \cdots$ are uniform.

(3) To generate a Markov chain $(\lambda, P)$, $\lambda$ being a law on $I$, we take a sequence $Y_1, Y_2, \ldots$ of uniform random variables on $[0,1]$. We define:

$$X_0 = j_r \text{ if } \sum_{n=1}^{r-1} \lambda(j_n) \leqslant Y_1 \leqslant \sum_{n=1}^{r} \lambda(j_n),$$

then,

$$X_{n+1} = G(X_n, Y_{n+1}) \ \ n = 0, 1, \ldots.$$