Exercise 1. Consider the following transition matrix:

$$\begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{pmatrix}$$

(1) Identify the communicating classes of this matrix.

(2) What are the closed classes?

Solution. (1) Let $I = \{0, 1, 2, 3, 4\}$. It is easy to see that $0 \nrightarrow j$ unless $j = 0, 4$, and $4 \nrightarrow j$ unless $j = 0, 4$. Thus $C_1 = \{0, 4\}$ is a communicating class. We also deduce that $C_2 = \{2\}$ is a communicating class. Since $1 \to 3$ and $3 \to 1$, we necessarily have that $C_3 = \{1, 3\}$ is also a communicating class.

(2) C_1 and C_2 are closed.

Exercise 2. Suppose that the probability it rains today is 0.3 if neither of the last two days was rainy, but 0.6 if at least one of the last two days was rainy. Let the weather on day n, W_n, be R for rain, or S for sun. W_n is not a Markov chain, but the weather for the last two days $X_n = (W_{n-1}, W_n)$ is a Markov chain with four states $\{RR, RS, SR, SS\}$.

(a) Find the transition matrix corresponding to X_n.

(b) What is the probability it will rain on Wednesday given that it did not rain on Sunday and Monday?

Solution. We write R for a rainy day, S for a sunny day and RR for the event $(W_{n-1}, W_n) = (R, R)$.

(a) We have, for $n \geq 1$,

$$\begin{aligned} \mathbb{P}(X_{n+1} = RR \mid X_n = RR) =& \mathbb{P}(W_{n+1} = R, W_n = R \mid W_n = R, W_{n-1} = R) \\ =& \mathbb{P}(W_{n+1} = R \mid W_n = R, W_{n-1} = R) = 0.6. \end{aligned}$$

A similar computation for the states RS, SR, SS gives

$$Q = \begin{pmatrix} 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 \\ 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 \end{pmatrix}$$

(b) We need to find the probabiloity p to go from $X_n = (S, S)$ to $X_{n+2} = (\cdot, R)$. By the total probability formula, we get

$$\begin{aligned} p = \mathbb{P}(W_{n+2} = R \mid X_n = SS) =& \mathbb{P}(W_{n+2} = R, W_{n+1} = R \mid X_n = SS) \\ &+ \mathbb{P}(W_{n+2} = R, W_{n+1} = S \mid X_n = SS) \\ =& \mathbb{P}(X_{n+2} = RR \mid X_n = SS) + \mathbb{P}(X_{n+2} = SR \mid X_n = SS) \\ =& Q^2(SS, SR) + Q^2(SS, RR) \\ =& 0.3 \cdot 0.7 + 0.3 \cdot 0.6 = 0.39. \end{aligned}$$

Exercise 3. A frog is jumping on the endpoints of a segment $[A, B]$ such that $p_{A,B} = \alpha$ and $p_{B,A} = \beta$, and the transition matrix is given by

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}.$$

Starting from point A, what is the probability that it will return to the same point after n jumps? In general, find the matrix P^n. Give conditions for the convergence $\lim_{n\to\infty} P^n =: P^\infty$ and compute this limit. How do you interpret the entries of the matrix P^∞.
Hint: diagonalize the transition matrix. The entries of P^n are linear combinations of λ_1^n and λ_2^n, where λ_1 and λ_2 are the eigenvalues of P.

Solution. The eigenvalues of P are the roots of the equation

$$\det(\lambda I_2 - P) = 0.$$

But we can write

$$\det(\lambda I_2 - P) = (\lambda - (1-\alpha))(\lambda - (1-\beta)) - \alpha\beta = \lambda^2 + \lambda(\alpha+\beta-2) + (1-\alpha-\beta).$$

The roots of this equation are given by

$$\lambda_1 = 1-\alpha-\beta, \quad \lambda_2 = 1.$$

Writing $D = \mathrm{diag}(\lambda_1, \lambda_2)$, there exists an invertible matrix Q such that $P = QDQ^{-1}$. Therefore, $P^n = QD^nQ^{-1}$ and so the coefficient $p_{AA}^{(n)}$ of the matrix P^n is a linear combination of λ_1^n and λ_2^n. Thus, there exists real numbers a_1 and b_1 such that

$$p_{AA}^{(n)} = a_1 + b_1(1-\alpha-\beta)^n.$$

Noticing that $p_{AA}^{(0)} = 1$ and $p_{AA}^{(1)} = p_{AA} = 1-\alpha$, we obtain, by a simple computation

$$a_1 = \frac{\beta}{\alpha+\beta}, \quad b_1 = 1 - a_1 = \frac{\alpha}{\alpha+\beta}.$$

We finally obtain:

$$p_{AA}^{(n)} = \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n.$$

Similarly, there exist constants a_2 and b_2 such that

$$p_{BB}^{(n)} = a_2 + b_2(1-\alpha-\beta)^n,$$

Using that $p_{BB}^{(0)} = 1$ and that $p_{BB}^{(1)} = 1-\beta$, we easily get

$$a_2 = \frac{\alpha}{\alpha+\beta}, \quad b_2 = 1 - a_2 = \frac{\beta}{\alpha+\beta}.$$

By the total probability formula, $p_{AA}^{(n)} + p_{AB}^{(n)} = 1$. We have likewise $p_{BA}^{(n)} + p_{BB}^{(n)} = 1$. The general form of the matrix P^n is thus given by

$$\begin{pmatrix} \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n & \frac{\alpha}{\alpha+\beta} - \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n \\ \frac{\beta}{\alpha+\beta} - \frac{\beta}{\alpha+\beta}(1-\alpha-\beta)^n & \frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta}(1-\alpha-\beta)^n \end{pmatrix}.$$

The convergence of the matrix P^n is garanteed if the condition $-1 < 1-\alpha-\beta \leq 1$ is satisfied. This is equivalent to $0 \leq \alpha + \beta < 2$. In other words, the matrix P^n converges unless we have $\alpha = \beta = 1$. The limiting matrix P^∞ is given by

$$P^\infty = \begin{pmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{pmatrix}.$$

We notice that the initial distribution (of X_0) is given by $\vec{p} := (p_A, p_B)$, so the distribution of X_n given by $\vec{p} \cdot P^n$ converges to

$$\vec{p} \cdot P^n \overset{n\to\infty}{\longrightarrow} \vec{p} \cdot P^\infty = (\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}).$$

Hence the distribution of X_n always converges to $(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta})$ independently of the initial distribution $\vec{p}$.

Exercise 4. Show that any finite transition matrix has at least one closed communicating class. Find an example of a transition matrix without any closed communicating class.

Solution. We proceed by induction on the size of the states space. We write: $P(n)$: "the statement is true for any Markov chain on I with $\text{Card}(I) \leqslant n$". $P(1)$ is evident. Suppose $P(n-1)$ and consider a Markov chain on $I = \{i_1, i_2, \ldots, i_n\}$. Let C the communicating class of i_n. If $C = I$ then $P(n)$ is true for I. If $C \neq I$, then there exists $j \in I \setminus C$ such that either $j \nrightarrow i_n$ or $i_n \nrightarrow j$. Without loss of generality, we suppose that $i_n \nrightarrow j$ and so $k \nrightarrow j \ \forall k \in C$. Let $C' = \{w : i_n \to w\}$ (so $C \subseteq C'$). We necessarily have:

(1) $j \notin C'$,

(2) if $k \in C'$ then $k \nrightarrow j$, and consequently if $k \in C'$ is such that $p_{k,k'} > 0$ then $k' \in C'$.

Hence the restriction of p to C' gives the transition probabilities of a Markov chain on C'. According to (1), we have that $\text{Card}(C') \leqslant n-1$. This implies by the induction hypothesis $P(n-1)$ that there exists a closed communicating class; this class belongs obviously to the initial chain.

Exercise 5. Let $(X_n)_{n\geq 0}$ be a Markov chain on a state space E and a transition matrix P. For each state $i \in E$, we write τ_i for the duration of a visit in i (if $X_0 = i$, τ_i is then the moment when the Markov chain leaves i for the first time).
determine the law of τ_i .

Solution.

$$\begin{aligned}
\mathbb{P}(\tau_i = k \mid X_0 = i) &= \mathbb{P}(X_1 = i, X_2 = i, \cdots, X_{k-1} = i, X_k \neq i \mid X_0 = i) \\
&= \mathbb{P}(X_2 = i, \cdots, X_{k-1} = i, X_k \neq i \mid X_0 = i, X_1 = i) \times \underbrace{\mathbb{P}(X_1 = i \mid X_0 = i)}_{P_{ii}} \\
&= P_{ii} \times \mathbb{P}(X_3 = i, \cdots, X_{k-1} = i, X_k \neq i \mid X_1 = i, X_2 = i) \times \mathbb{P}(X_2 = i \mid X_1 = i) \\
&= P_{ii}^{k-1}(1 - P_{ii}), \ k \in \mathbb{N}^*.
\end{aligned}$$

Therefore, τ_i is distributed according to a geometric law of parameter $(1 - P_{ii})$.