Exercise 1. Let X and Y be two independent random variables following a Poisson distribution with respective parameters λ and μ.

(1) Find the law of $X+Y$.

(2) Find the conditional law of X knowing $X+Y=n$.

Solution. Remark: The symbol $\perp\!\!\!\perp$ means independence.

(1)

$$\begin{aligned}
\mathbb{P}(X+Y=n) &= \sum_{m=0}^{\infty} \mathbb{P}(X+Y=n, Y=m) \\
&= \sum_{m=0}^{n} \mathbb{P}(X=n-m, Y=m) \\
&= \sum_{m=0}^{n} \mathbb{P}(X=n-m)\mathbb{P}(Y=m) \qquad (X \perp\!\!\!\perp Y) \\
&= \sum_{m=0}^{n} e^{-\lambda}\frac{\lambda^{n-m}}{(n-m)!}e^{-\mu}\frac{\mu^m}{m!} \\
&= e^{-(\lambda+\mu)}\frac{1}{n!}\sum_{m=0}^{n}\frac{n!}{m!\,(n-m)!}\lambda^{n-m}\mu^m \\
&= e^{-(\lambda+\mu)}\frac{(\lambda+\mu)^n}{n!},
\end{aligned}$$

using the Newton binomial formula. So $X+Y \sim \text{Poisson}(\lambda+\mu)$.

(2)

$$\begin{aligned}
\mathbb{P}(X=k \mid X+Y=n) &= \frac{\mathbb{P}(\{X=k\}\cap\{X+Y=n\})}{\mathbb{P}(X+Y=n)} \qquad (\text{for } k \leqslant n,\ 0 \text{ otherwise}) \\
&= \frac{\mathbb{P}(X+Y=n \mid X=k)\,\mathbb{P}(X=k)}{\mathbb{P}(X+Y=n)} \\
&= \frac{\mathbb{P}(Y=n-k)\,\mathbb{P}(X=k)}{\mathbb{P}(X+Y=n)} \qquad (X+Y \sim \text{Poisson}(\lambda+\mu)) \\
&= \binom{n}{k}\frac{\lambda^k}{(\lambda+\mu)^k}\frac{\mu^{n-k}}{(\lambda+\mu)^{n-k}}.
\end{aligned}$$

So

$$\mathbb{P}(X=k \mid X+Y=n) = \binom{n}{k}\left(\frac{\lambda}{\lambda+\mu}\right)^k\left(\frac{\mu}{\lambda+\mu}\right)^{n-k} \qquad k=0,1,2,\ldots,n.$$

Which implies that $X \mid \{X+Y=n\} \sim \text{Bin}(n,p)$ where $p=\frac{\lambda}{\lambda+\mu}$.

Exercise 2. Let X and Y be two independent exponential random variables with respective parameters λ and μ. What is the probability that $Y \leq X$?

Solution. By direct computation we obtain

$$\begin{aligned}
\mathbb{P}(Y \leqslant X) &= \int_0^\infty \mathbb{P}(Y \leqslant X \mid X = x) f_X(x)dx \\
&= \int_0^\infty \mathbb{P}(Y \leqslant x) f_X(x)dx \qquad (X \perp\!\!\!\perp Y) \\
&= \int_0^\infty F_Y(x) f_X(x)dx \\
&= \int_0^\infty \left(1 - e^{-\mu x}\right) \lambda e^{-\lambda x}dx \\
&= 1 - \int_0^\infty \lambda e^{-(\lambda+\mu)x}dx \\
&= \frac{\mu}{\mu + \lambda}.
\end{aligned}$$

Exercise 3. Let $X_1, X_2, \ldots, X_n$ be independent exponential random variables with parameters $\mu_1, \mu_2, \ldots, \mu_n$. Show that the random variable $Z = \min\{X_1, X_2, \ldots, X_n\}$ is again exponentially distributed and find its parameter.

Solution.

$$\begin{aligned}
\mathbb{P}(Z > x) &= \mathbb{P}(\{X_1 > x\} \cap \{X_2 > x\} \cap \ldots \cap \{X_n > x\}) \\
&= \prod_{i=1}^n \mathbb{P}(X_i > x), \qquad (X_i \perp\!\!\!\perp X_j \text{ for } i \neq j) \\
&= \prod_{i=1}^n e^{-\mu_i x} \\
&= e^{-\sum_{i=1}^n \mu_i x} \\
&\Rightarrow F_Z(x) = 1 - e^{-\lambda x} \qquad \left(\lambda = \sum_i^n \mu_i\right) \\
&\Rightarrow Z \sim \operatorname{Exp}(\lambda).
\end{aligned}$$

Exercise 4. (**Memorylessness**) A random variable X is called *memorylessness* if $\forall s, t \geq 0$

$$\mathbb{P}\{X \geq t + s \mid X \geq s\} = \mathbb{P}\{X \geq t\}.$$

(1) Show that an exponential random variable has this property.

(2) Show that there is no other continuous random variable having this property.

Solution. (1) If $X \sim \operatorname{Exp}(\lambda)$:

$$\begin{aligned}
\mathbb{P}(X \geqslant t + s \mid X \geqslant s) &= \frac{\mathbb{P}(\{X \geqslant t + s\} \cap \{X \geqslant s\})}{\mathbb{P}(X \geqslant s)} \\
&= \frac{\mathbb{P}(X \geqslant t + s)}{\mathbb{P}(X \geqslant s)} \qquad (s, t \geqslant 0) \\
&= \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} \\
&= e^{-\lambda t} \\
&= \mathbb{P}(X \geqslant t).
\end{aligned}$$

(2) In order to show that exponential random variables are the only memoryless continuous random variables, we have to show that $\bar{F}(x) = \mathbb{P}(X \geqslant x)$ uniquely verifies the relation $\bar{F}(x+y) = \bar{F}(x)\bar{F}(y)$. Indeed

$$\mathbb{P}(X \geqslant x+y) = \mathbb{P}(X \geqslant x+y \mid X \geqslant x)\,\mathbb{P}(X \geqslant x) = \mathbb{P}(X \geqslant y)\,\mathbb{P}(X \geqslant x)$$

which implies

$$\begin{aligned}
&\bar{F}(0) = \bar{F}(0+0) = \bar{F}(0)^2 \Rightarrow \bar{F}(0) = 0 \text{ or } 1,\\
&\qquad \text{if } \bar{F}(0) = 0 : \bar{F}(x) = 0 \ \forall x > 0 \text{ since } \bar{F}(x) = \bar{F}(x+0) = \bar{F}(x)\bar{F}(0) = 0,\\
&\qquad \Rightarrow \bar{F}(0) = 1,\\
&\bar{F}(1) = \bar{F}(1/2 + 1/2) = \bar{F}(1/2)^2 \geqslant 0 \Rightarrow \bar{F}(1) = \alpha \geq 0,\\
&\bar{F}(n) = \bar{F}(1 + 1 + \ldots + 1) = \bar{F}(1)^n = \alpha^n, \qquad (n \in \mathbb{N}),\\
&\bar{F}(1) = \bar{F}(1/n + \ldots + 1/n) = \bar{F}(1/n)^n \Rightarrow \bar{F}(1/n) = \alpha^{1/n}, \qquad (n \in \mathbb{N}^*),\\
&\qquad \Rightarrow \bar{F}(m/n) = \alpha^{m/n}.
\end{aligned}$$

Then by the continuity of the random variable and the density of rationals in the reals we have $\bar{F}(x) = \alpha^x \ \forall x \geqslant 0$. As $\bar{F}(x)$ is a probability, we find that $f(x) = e^{-\lambda x}$ for $\lambda > 0$. Note that we have to remove the case where $\alpha = 0$, because otherwise the density would be a Dirac point mass in zero ($\mathbb{P}(X = 0) = 1$), which is obviously not continuous. Moreover, note that we could also have studied $\log \bar{F}(x)$ to deduce the result.

Exercise 5. Let the distance driven until failure of a new car battery be modeled by an exponential distribution with mean value 20000 kilometers. Somebody wants to go on a 10000 kilometers trip. We know that the car was used during k kilometers before (distance driven without failure since the last battery change).

(1) What is the probability that it will arrive at destination without battery failure?

(2) How does this probability change if we do not assume an exponential distribution?

Solution. (1) X denotes the life of the battery which is $\sim \operatorname{Exp}\left(\frac{1}{20000}\right)$. We compute

$$\begin{aligned}
&\mathbb{P}(X \geqslant 10000 + k \mid X \geqslant k)\\
&= \mathbb{P}(X \geqslant 10000) \qquad\qquad \text{(memorylessness)}\\
&= 1 - F_X(10000)\\
&= e^{-\frac{10000}{20000}}\\
&= \frac{1}{\sqrt{e}}.
\end{aligned}$$

(2) In this case we need to compute

$$\begin{aligned}
&\mathbb{P}(X \geqslant 10000 + k \mid X \geqslant k) \\
&= \frac{\mathbb{P}(\{X \geqslant 10000 + k\} \cap \{X \geqslant k\})}{\mathbb{P}(X \geqslant k)} \\
&= \frac{\mathbb{P}(X \geqslant 10000 + k)}{\mathbb{P}(X \geqslant k)} \\
&= \frac{1 - F_X(10000 + k)}{1 - F_X(k)}.
\end{aligned}$$

where F_X is the cumulative distribution function of X.

Remark: In point (1), we did not need to know k to compute this probability (thanks to the memorylessness). Here, the number k should be known to finalize the computations.

Exercise 6. Let X be a discrete random variable such that

$$\mathbb{P}\{X = n\} = \frac{2}{3^n} \ \forall n \in \mathbb{N} \setminus \{0\}.$$

We define the random variable Y as follow: knowing $X = n$, Y takes values n or $n+1$ with equal probability.

(1) Compute $\mathbb{E}(X)$.

(2) Compute $\mathbb{E}(Y|X = n)$ and deduce $\mathbb{E}(Y|X)$, then $\mathbb{E}(Y)$.

(3) Compute the joint law of (X, Y).

(4) Compute the marginal law of Y.

(5) Compute $\mathbb{E}(X|Y = i)$ ($\forall i \in \mathbb{N} \setminus \{0\}$) and deduce $\mathbb{E}(X|Y)$.

(6) Compute the Covariance of X and Y.

Solution. (1) Using the derivative of the geometrical series we get

$$\mathbb{E}(X) = \sum_{n=1}^{\infty} n \cdot \frac{2}{3^n} = \frac{2}{3} \sum_{n=1}^{\infty} n \left(\frac{1}{3}\right)^{n-1} = \frac{3}{2}.$$

(2)

$$\mathbb{E}(Y \mid X = n) = n/2 + (n+1)/2 = n + 1/2,$$

so

$$\mathbb{E}(Y \mid X) = X + 1/2,$$

implying

$$\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y \mid X)) = 3/2 + 1/2 = 2.$$

(3)
- $\mathbb{P}(X = n, Y = m) = 0$ if $m \neq n$ and $m \neq n + 1$.
- $\mathbb{P}(X = n, Y = n) = \mathbb{P}(Y = n \mid X = n)\, \mathbb{P}(X = n) = \frac{1}{2} \cdot \frac{2}{3^n} = \frac{1}{3^n}$.
- $\mathbb{P}(X = n, Y = n + 1) = \mathbb{P}(Y = n + 1 \mid X = n)\, \mathbb{P}(X = n) = \frac{1}{3^n}$.

(4) Let us treat the two cases

$(n=1)$: $\mathbb{P}(Y=1)=\mathbb{P}(Y=1,X=1)=1/3$.

$(n\geqslant 2)$:

$$\begin{aligned}\mathbb{P}(Y=n) &= \sum_{m=0}^{\infty}\mathbb{P}(X=m,Y=n)\\ &= \mathbb{P}(X=n,Y=n)+\mathbb{P}(X=n-1,Y=n)\\ &= \frac{1}{3^n}+\frac{1}{3^{n-1}}=\frac{4}{3^n}.\end{aligned}$$

(5) Again we treat two cases

$(i=1)$: $\mathbb{E}(X\mid Y=1)=1$. since $\mathbb{P}(X=1\mid Y=1)=1$.

$(i>1)$: We have

$$\mathbb{P}(X=i\mid Y=i)=\frac{\mathbb{P}(X=i,Y=i)}{\mathbb{P}(Y=i)}=\frac{1/3^i}{4/3^i}=\frac{1}{4},$$
$$\mathbb{P}(X=i-1\mid Y=i)=\frac{\mathbb{P}(X=i-1,Y=i)}{\mathbb{P}(Y=i)}=\frac{1/3^{i-1}}{4/3^i}=\frac{3}{4}.$$

Hence,

$$\begin{aligned}\mathbb{E}(X\mid Y=i) &= \sum_{k=1}^{\infty}k\mathbb{P}(X=k\mid Y=i)\\ &= i\mathbb{P}(X=i\mid Y=i)+(i-1)\mathbb{P}(X=i-1\mid Y=i)\\ &= \frac{4i-3}{4}.\end{aligned}$$

So that $\mathbb{E}(X\mid Y)=\frac{4Y-3}{4}\mathbb{1}_{\{Y>1\}}+\mathbb{1}_{\{Y=1\}}$.

(6) $\mathrm{Cov}(X,Y)=\mathbb{E}(XY)-\mathbb{E}(X)\mathbb{E}(Y)=\mathbb{E}(XY)-3$ and

$$\begin{aligned}\mathbb{E}(XY) &= \mathbb{E}(\mathbb{E}(XY\mid X))=\mathbb{E}(X\,\mathbb{E}(Y\mid X))\\ &= \mathbb{E}(X(X+1/2))=\mathbb{E}\left(X^2\right)+1/2\,\mathbb{E}(X).\end{aligned}$$

We need to compute $\mathbb{E}(X^2)$

$$\begin{aligned}\mathbb{E}\left(X^2\right) &= \textstyle\sum_{n=1}^{\infty}n^2\frac{2}{3^n}=\sum_{n=1}^{\infty}2\left(n(n-1)+n\right)\left(\frac{1}{3}\right)^n\\ &= \textstyle\frac{3}{2}+\frac{2}{9}\sum_{n=2}^{\infty}n(n-1)\left(\frac{1}{3}\right)^{n-2}=\frac{3}{2}+\frac{3}{2}=3,\end{aligned}$$

which implies $\mathrm{Cov}(X,Y)=\frac{3}{4}$.