Exercise 1. In each of the following cases, compute $\lim_{t\to\infty} \mathbb{P}(X_t = 2 \mid X_0 = 1)$ for the Markov chain $(X_t)_{t\geq 0}$ with the given Q-matrix on $\{1,2,3,4\}$:

$$(a) \begin{pmatrix} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \qquad (b) \begin{pmatrix} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(c) \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 2 & -2 \end{pmatrix} \qquad (d) \begin{pmatrix} -2 & 1 & 0 & 1 \\ 0 & -2 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution. (a) We need to find the corresponding stationary distribution π verifying

$$\pi Q = 0, \quad \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1.$$

So we get

$$-2\pi_1 + \pi_4 = 0,\ \pi_1 - \pi_2 = 0,\ \pi_1 + \pi_2 - \pi_3 = 0,\ \pi_3 - \pi_4 = 0.$$

Solving this, we get $\pi = (\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3})$. So we have that $\lim_{t\to\infty} \mathbb{P}(X_t = 2 \mid X_0 = 1) = \pi_2 = \frac{1}{6}$.

(b) Since 4 is an absorbing state, we have that $\lim_{t\to\infty} \mathbb{P}(X_t = 2 \mid X_0 = 1) = 0$ in this case and $\lim_{t\to\infty} \mathbb{P}(X_t = 4 \mid X_0 = 1) = 1$.

(c) We need to find the stationory distribution π corresponding to 1 and 2 using their sub-matrix

$$Q_1 := \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

We get easily that $\pi = (\frac{1}{2}, \frac{1}{2})$ and so $\lim_{t\to\infty} \mathbb{P}(X_t = 2 \mid X_0 = 1) = \frac{1}{2}$.

(d) First we find the proportion of time spent at 2 by computing $\pi Q_2 = 0$ where

$$Q_2 := \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}.$$

We get $\pi = (\pi_1, \pi_2) = (\frac{1}{3}, \frac{2}{3})$ where π_1 corresponds to state 2 in this case. So we get

$$\lim_{t\to\infty} \mathbb{P}(X_t = 2 \mid X_0 = 1) = \pi_1 \times \mathbb{P}(X_t \text{ hits } 2 \mid X_0 = 1) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}.$$

Exercise 2. Customers arrive at a single-server queue in a Poisson stream of rate λ. Each customer has a service requirement distributed as the sum of two independent exponential random variables of parameter μ. Service requirements are independent of one another and of the arrival process.

(a) Write down the generator matrix Q of a continuous-time Markov chain which models this, explaining what the states of the chain represent.

(b) Verify that the stationary distribution at state n is of the form $\pi_n = ax_1^n + bx_2^n$ with specific x_1 and x_2 and deduce that the chain is positive recurrent if and only if $\lambda/\mu < \frac{1}{2}$.

Solution. (a) In this chain, a customer will need 2 times exponential times with rate μ as service time. This can be seen as two customers that need each an exponential time with rate μ as a service time. So we can describe the chain as follows: If we are at state n, we go to state $n+2$ after an exponential time with rate λ and to state $n-1$ after an exponential time of rate μ. So:

$$Q_{n,n+2} = \lambda,\ Q_{n,n-1} = \mu,\ Q_{n,n} = -\lambda - \mu,\ Q_{n,j} = 0\ \forall j \neq n-1, n, n+2.$$

(b) Conditioning on the last step before reaching state n, the stationary distribution π must verify

$$\pi_n = \frac{\lambda}{\lambda+\mu}\pi_{n-2} + \frac{\mu}{\lambda+\mu}\pi_{n+1}.$$

This gives the following characteristic equation

$$(\lambda+\mu)x^2 = \lambda + \mu x^3 \iff (x-1)(\mu x^2 - \lambda x - \lambda) = 0$$

The roots of the second degree polynomial term are

$$x_1 = \frac{\lambda - \sqrt{\lambda^2 + 4\mu\lambda}}{2\mu},\quad x_2 = \frac{\lambda + \sqrt{\lambda^2 + 4\mu\lambda}}{2\mu}.$$

So π_n is of the form

$$ax_1^n + bx_2^n + c,$$

for real numbers a, b, c. Since $\sum_{n\geq 0}\pi_n = 1$, we deduce that $c = 0$. On the other hand, since $x_1 < 0$, we deduce that $b > 0$ to ensure that $\pi_n > 0$. We also need to verify that $x_2 < 1$ in order to garantee a convergent serie $\sum_{n\geq 0}\pi_n$.
This is equivalent to

$$\frac{\lambda + \sqrt{\lambda^2 + 4\mu\lambda}}{2\mu} < 1, \iff 2\mu - \lambda > \sqrt{\lambda^2 + 4\mu\lambda} \iff 4\mu^2 > 8\mu\lambda \iff \mu > 2\lambda.$$

So a stationary distribution of the system exists if and only if $\mu > 2\lambda$.

Exercise 3. Let $\{X(t) \mid t \in \mathbb{R}^+\}$ be a Markov process with n states $\{1, 2, \cdots, n\}$ and generator:

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda_2 & \lambda_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\lambda_3 & \lambda_3 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -\lambda_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\lambda_{n-1} & \lambda_{n-1} \\ \lambda_n & 0 & 0 & 0 & \cdots & 0 & -\lambda_n \end{pmatrix}.$$

(i). Find the stationary distribution of the process

(ii). Give an intuitive explanation of the result.

Solution. Starting in a given state i, we see that we have a cyclic path, we go deterministically to $i+1$ if $i \neq n$ and in 1 if $i = n$, spending a exponentially distributed time with parameter λ_i in each position.
Intuitively, the stationary distribution in i is the mean proportion of time spent in i during a full cycle, that is

$$\pi_i = \frac{\frac{1}{\lambda_i}}{\sum_{j=1}^{n} \frac{1}{\lambda_k}}.$$

We check now that it is really a stationary distribution. Clearly,

$$\sum_{j=1}^{n} \pi_j = 1.$$

Note that $\pi Q = 0$, since

$$\begin{cases} \lambda_i \pi_i = \lambda_{i+1}\pi_{i+1}, \, si \; i \neq n \\ \lambda_n \pi_n = \lambda_1 \pi_1 \end{cases}$$

Note that unicity is obtained by a direct resolution of the system.

Exercise 4. Let $\{X(t) \mid t \in \mathbb{R}^+\}$ be a Markov process with n states $\{1, 2, \cdots, n\}$ and generator:

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ \lambda_2\mu_2 & -\lambda_2 & \lambda_2(1-\mu_2) & 0 & \cdots & 0 & 0 \\ \lambda_3\mu_3 & 0 & -\lambda_3 & \lambda_3(1-\mu_3) & \cdots & 0 & 0 \\ \lambda_4\mu_4 & 0 & 0 & -\lambda_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{n-1}\mu_{n-1} & 0 & 0 & 0 & \cdots & -\lambda_{n-1} & \lambda_{n-1}(1-\mu_{n-1}) \\ \lambda_n & 0 & 0 & 0 & \cdots & 0 & -\lambda_n \end{pmatrix}.$$

(i). Guess the expression $\pi_1 = \lim_{t\to\infty} P[X(t) = 1]$.

(ii). Find the stationary distribution of the process.

Solution. At every state i except 1 and n, we jump to the next state with probability $1-\mu_i$, or we return to 1 with probability μ_i, we deterministically return to 1 in position n and we jump in 2 when we are in 1. As before, we infer that the stationary distribution in 1 is equal to the mean time spent in 1, divided by the duration of a return in 1—duration of a cycle. Let us denote C the duration of a cycle, then

$$\begin{aligned} \mathrm{E}[C] &= \sum_{k=2}^{n} \mathrm{E}\left[C \mid \textit{the last state visited before 1 is } k\right] \mathbb{P}\{\textit{the last state visited before 1 is } k\} \\ &= \sum_{k=2}^{n} \left(\tfrac{1}{\lambda_1} + \cdots + \tfrac{1}{\lambda_k}\right) \times (1-\mu_2)\cdots(1-\mu_{k-1})\mu_k, \end{aligned}$$

and so

$$\pi_1 = \frac{\frac{1}{\lambda_1}}{\mathrm{E}[C]}.$$

With the same methodology, we find π_i, $2 \le i \le n$. Let τ_i be the time spent in i during a cycle, then

$$\begin{aligned}\mathrm{E}[\tau_i] &= \mathrm{E}\left[\tau_i | i \textit{ hit during the cycle}\right] \mathbb{P}\{i \textit{ hit during the cycle}\} \\ &= \tfrac{1}{\lambda_i}(1-\mu_2)\cdots(1-\mu_{i-1}),\end{aligned}$$

we used that

$$\mathrm{E}\left[\tau_i | i \textit{ is not hit during the cycle}\right] = 0.$$

Thus,

$$\pi_i = \frac{\mathrm{E}[\tau_i]}{\mathrm{E}[C]}.$$

Exercise 5. Let $\{X(t) \mid t \in \mathbb{R}^+\}$ be an irreducible Markov process on a finite space of n states, with generator Q. Let us take λ so that $\lambda > \max_i\{-Q_{ii}\}$ and define the matrix:

$$P = I + \frac{1}{\lambda}Q \qquad \text{(where } I \text{ is the identity matrix).}$$

(i). Show that P is a transition matrix, and that its stationary distribution is identical to the one of Q.

(ii). Let $\{N(t) \mid t \in \mathbb{R}^+\}$ be a Poisson process with parameter λ and let $\{Y_k \mid k \in \mathbb{N}\}$ be a Markov chain with matrix P, independent from $\{N(t)\}$. Let $T_0 = 0$, and $T_1, T_2, \cdots$ denote the arrival times in the Poisson process.

We define the process $\{Z(t) \mid t \in \mathbb{R}^+\}$ as follows:

$$Z(t) = Y_n \qquad \forall t \in [T_n, T_{n+1}[.$$

Show that $\{Z(t)\}$ is a Markov process with generator Q.

Solution. (i). Let π be the stationary distribution of Q, then

$$\pi P = \pi + \underbrace{\tfrac{1}{\lambda}\pi Q}_{=0} = \pi.$$

As there exists a unique solution (to a multiplicative constant) satisfying $xP = x$, π is the stationary distribution.
(ii). $\{Z(t)\}$ is a Markov process, since,

- $\mathbb{P}\{Z(t+s) = j | Z(u), 0 \le u \le s\} = \mathbb{P}\{Z(t+s) = j | Z(s)\}$, since the Poisson process N is Markovian, its arrivals between s and $s+t$ do not depend on the history of the process before s. Moreover, $\{Y_n\}$ satisfy the Markov property and thus the future transitions depend only the the current one s.

- $\mathbb{P}\{Z(t+s)=j|Z(s)=i\}=\mathbb{P}\{Z(t)=j|Z(0)=i\}$:

$$\begin{aligned}\mathbb{P}\{Z(t+s)=j|Z(s)=i\} =& \sum_k \mathbb{P}\{Z(t+s)=j|Z(s)=i,\ k \textit{ arrivals from } N \textit{ on } [s,s+t]\} \\ & \times \underbrace{\mathbb{P}\{N(t+s)-N(s)=k|Z(s)=i\}}_{=\mathbb{P}\{N(t)=k|Z(0)=i\}} \\ =& \sum_k (P^k)_{ij}\mathbb{P}\{N(t)=k|Z(0)=i\} \\ =& \mathbb{P}\{Z(t)=j|Z(0)=i\}.\end{aligned}$$

Let us show now that it is a Markov process with generator Q. We show that the transition matrix of $\{Z(t)\}$ can be expressed as the exponential of Q,

$$\begin{aligned}L_{ij}(t) :=& \mathbb{P}\{Z(t)=j|Z(0)=i\} \\ =& \sum_{k=0}^{\infty} \mathbb{P}\{Z(t)=j|Z(0)=i,\ k \textit{ arrivals of } N \textit{ on } [0,t]\} \\ & \times \mathbb{P}\{N(t)=k|Z(0)=i\} \\ =& \sum_{k=0}^{\infty} (P^k)_{ij}\exp\{-\lambda t\}\frac{(\lambda t)^k}{k!}.\end{aligned}$$

So that,

$$L(t)=\exp\{-\lambda I t\}\sum_{k=0}^{\infty}\frac{(\lambda t P)^k}{k!}=\exp\{(\lambda P-\lambda I)t\},$$

and $\lambda P-\lambda I=\lambda(I+\frac{1}{\lambda}Q)-\lambda I=Q$, thus

$$L(t)=\exp\{Qt\}.$$

Exercise 6. Show that the renewal function $R(t)$ satisfy a renewal equation, and specify the corresponding function $g(t)$.

Solution. We could get a renewal equation for $R(t)$ by conditioning on the first time S_1 after S_0. However, here, since $R(t)=\sum_{n\geq 0}F^{(n)}(t)$, we directly get that

$$\begin{aligned}(R*F)(t) &= \sum_{n\geq 0}(F^{(n)}*F)(t)=\sum_{n\geq 0}F^{(n+1)}(t), \\ &= \sum_{n\geq 1}F^{(n)}(t)=\sum_{n\geq 0}F^{(n)}(t)-F^{(0)}(t),\end{aligned}$$

and so

$$R(t)=F^{(0)}(t)+(R*F)(t).$$

with, $g(t)=F^{(0)}(t)=1_{\{t\geq 0\}}$.
This also shows that $t\geq 0$, $(R*F)(t)=R(t)-1$, which was used during the lecture in the proof of the theorem giving the distribution of the duration of life L of a transitive renewal process.

Exercise 7. Let $S_1, S_2, \ldots$ be the successive times at which cars cross a certain fixed position on the highway. We assume that the intervals of time $W_1, W_2, \ldots$ between each renewal are i.i.d. with cumulative distribution $F(\cdot)$. Suppose that at time $t = 0$, a pedestrian arrives at this fixed position, and wants to cross the road. Assume that he needs τ units of time to cross it. Let L be the time that the pedestrian has to wait before starting to cross the road.

(a) Find the distribution of L and its expectation.

(b) Same questions if we assume that the arrivals of cars follow a Poisson process with parameter λ.

Solution. (a). The pedestrian starts to cross the road at $L = S_n$ if and only if $W_1 \leq \tau, \ldots, W_n \leq \tau$ and $W_{n+1} > \tau$. So that, L is the duration of life of a renewal $\{\tilde{S}_n\}$ having its n^{th} interval of time given by

$$\tilde{W}_n = \begin{cases} W_n, & si\ W_n \leq \tau, \\ +\infty, & otherwise. \end{cases}$$

Thus, the distribution of $\tilde{W}_n$ is given by

$$\tilde{F}(t) = \begin{cases} F(t), & if\, t \leq \tau, \\ F(\tau), & if\, t > \tau. \end{cases}$$

We deduce that

$$\mathbb{P}\{L \leq t\} = \{1 - \tilde{F}(\infty)\}\tilde{R}(t) = \{1 - F(\tau)\}\tilde{R}(t),$$

where $\tilde{R}(t) = \sum_{n \geq 0} \tilde{F}^{(n)}(t)$.

(b). The mean waiting time of the pedestrian is

$$\mathrm{E}[L] = \frac{1}{1 - F(\tau)} \int_0^\tau \{F(\tau) - F(t)\}dt.$$

More precisely, if the arrivals of the cars follow a Poisson process with parameter λ, then

$$\mathrm{E}[L] = \frac{1}{\lambda}\big(\exp\{\lambda\tau\} - 1\big) - \tau.$$