Continuous time Markov Chains

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Q-matrices

We first (and to avoid complications) define Markov chains on a finite state space I (so $|I| < \infty$). The Markov chain will be defined via a

- an initial probability $\lambda_i \in [0,1]\ \forall i \in I$ and
- A jump rate $q_i \in [0, \infty)\ \forall i \in I$ and
- Jump probabilities π_{ij} for $i, j \in I$.

A corresponding Markov chain $(X_t)_{t \geq 0}$ will satisfy

- $P(X_0 = i) = \lambda_i\ \forall i \in I$ and
- At every visit to state i, the Markov chain will stay there foran $\mathcal{E}xp(q_i)$ amount of time independent of the Markov chain's history before jumping to i. If $q_i = 0$ it is understood that the chain will remain at i forever once it visits it.
- After staying at state i an $\mathcal{E}xp(q_i)$ amount of time, the chain jumps to state j with probability π_{ij} for $i, j \in I$. If $q_i \neq 0$, then $\pi_{ii} = 0$. If $q_i = 0$ take $\pi_{ii} = 1$.

Two constructions

(I) We can associate to each state i $i.i.d$ sequences of $\mathcal{E}xp(q_i)$ r.v.s W_k^i $k \geq 1$ and i.i.d. r.v.s on $I \backslash \{i\}$, V_k^i $k \geq 1$ with distribution π_{ij} independent of other sequences and the value X_0 , so that on the k'th visit to state i, the chain rests time W_k^i before jumping to state $j = V_k^i$.

(ii) A second construction is to have independent (and independent of X_0) Poisson processes of rate q_{ij}, U_{ij} for each ordered pair (i, j) of distinct sites where $q_{ij} = q_i \pi_{ij}$ for $i \neq j$. We construct the Markov chain X by ruling that after having jumped to state i at (random) time t, the process stays at i until $\tau_t = \inf\{s > t : s$ is a jump time for some $U_{ij}\}$ at which point X jumps to the associated j.

Q matrices

Definition:

Given q_i $\forall i \in I$ and π_{ij} for $i, j \in I$, we define the $Q - matrix$ to be

$$\text{for } i \neq j: \; q_{ij} = q_i \pi_{ij}; q_{ii} = -q_i$$

So apart from knowing π_{ij} for i with $q_i = 0$, knowing the matrix Q tells us the matrix Π and the vector $\{q_i\}_{i \in I}$ and vice versa.
The matrix Q satisfies.

- $\forall i \; q_{ij} \geq 0$ unless $i = j$ and $q_{ii} < 0$ unless $q_{ii} = 0$
- $\forall i \sum_j q_{ij} = 0$,
- $\pi_{ij} = \frac{q_{ij}}{q_i} = \frac{-q_{ij}}{q_{II}}$

Any matrix on $I \times I$ satisfying the first two conditions above is called a Q-matrix.

Q matrices

- $\forall i q_{ij} \geq 0$ unless $i = j$ and $q_{ii} < 0$ unless $q_{ii} = 0$
- $\forall i \sum_j q_{ij} = 0$,
- $\pi_{ij} = \frac{q_{ij}}{q_i} \;=\; \frac{-q_{ij}}{q_{ll}}$ if defined.

We talk of (λ, Q) Markov chains (in continuous time) just as we talked of (λ, P) Markov chains in discrete time.

Jump Chains

Given a (λ, Q) Markov chain $(X_t)_{t\geq 0}$, the associated jump chain $(Y_n)_{n\geq 0}$ is defined recursivly as follows:
Let $J_0 = 0$ and for $k \geq 0, J_{k+1} = \inf\{t > J_k : X_t \neq X_{J_k}\}$. Let $S_n = J_n - J_{n-1}, n \geq 1$ and let $Y_n = X_{J_n}$ $n \geq 0$

Theorem

Under the above conditions $(Y_n)_{n\geq 0}$ *is a* (λ, Π) *Markov chain and given* $(Y_0, Y_1, \cdots Y_n, \cdots)$, *the random variables* S_k $i \geq 1$ *are independent exponential random variables with parameters* $q_{Y_{n-1}}$.

Semi Group

Definition:

Given a Q-matrix Q, the semigroup for the associated Markov chain is

$$P_{ij}(t) = P\left(X_t = j | X_0 = i\right) \text{ for } t \geq 0.$$

Theorem

For each $s, t \geq 0, P(t+s) = P(t)P(s) = P(s)P(t)$ *and for* X *a* (λ, Q) *Markov chain and* n *an integer* $0 < t_1 < t_2 \cdots < t_n$ *and* $i_k \in I$*,*

$$P(X_0 = i_0, X_{t_1} = i_1, \cdots X_{t_n} = i_n) = \lambda(i_0) \prod_{k=1}^{n} P_{i_{k-1} i_k}(t_k - t_{k-1})$$

Corollary

For a right continuous process $(X_t)_{t\geq 0}$, *the following are equivalent*

- X *is a* (λ, Q) *Markov chain*
- $\forall n$ *an integer* $0 < t_1 < t_2 \cdots < t_n$ *and* $i_k \in I$,
  $P(X_0 = i_0, X_{t_1} = i_1, \cdots X_{t_n} = i_n) = \lambda(i_0) \prod_{k=1}^{n} P_{i_{k-1} i_k}(t_k - t_{k-1})$

Backward/Forward equations

Theorem

For a Q-matrix defined on finite I and P the associated semigroup, we have That P is differentiable as a function of t and

$$P'(t) = QP(t) = P(t)Q$$

The first equation $P'(t) = QP(t)$ is called the *backward* equation, the equation $P'(t) = P(t)Q$ is called the *forward* equation.

Proof Outline

As usual we consider $\frac{P(t+h)-P(t)}{h}$ as h tends to zero. By the semigroup property it is equal to

$$P(t)\frac{P(h)-I_d}{h} = \frac{P(h)-I_d}{h}P(t)$$

But it is easy to check that $\frac{P(h)-I_d}{h}$ tends to Q as h becomes small. A key element that is not available for I infinite, is that there exists a constant $c < \infty$ so that the probability of two jumps by the markov chain in time interval $[0, h]$ is bounded above by ch^2. For $-\frac{P(t-h)-P(t)}{h}$, it is more of less the same, equalling

$$P(t-h)\frac{P(h)-I_d}{h} = \frac{P(h)-I_d}{h}P(t-h)$$

the same limit pertains as it is easy to see that $s \to P(s)$ is continuous.

Some General theory

Consider any matrix Q defined on (finite) set I. It is easy to see that for any positive integer r and $i, j \in I$, $|Q^r_{ij}| \leq (|I| \max_{uv} |Q_{uv}|^r$. Thus the matrix

$$exp(Q) = \sum_{n \geq 0} Q^n/n!$$

Is well defined.

Theorem

If Q is a matrix on $I \times I$ for I finite, then $P(t) = exp(tQ)$ satisfies

(i) $\forall s, t \geq 0 \; P(t+s) = P(t)P(s)$

(ii) $P(.)$ *is the unique solution of* $P'(t) = P(t)Q, \; P(0) = I_d$

(iii) $P(.)$ *is the unique solution of* $P'(t) = QP(t) \; P(0) = I_d$

(iv) $P(t)^{(k)}|_{t=0} = Q^k$.

Proof. For (i) the result is just algebra and the binomial formula (as we have infinite radius of convergence). For (ii) and (iii) again, as we have infinite radius of convergence, we can differentiate term by term (in t) to get the desired differential equation. To show it is unique, We consider (ii) and suppose that $M(t)$ is another solution (necessarily differentiable). Then consider $A(t) = M(t)e^{-Qt}$,

$$A'(t) = M'(t)e^{-Qt} - M(t)Qe^{-Qt} = (M'(t) - M(t)Q)e^{-Qt} = 0.$$

So $A(t)$ is a constant. But $A(0) = I_d$ and we have the result.

For (iv) we have by induction that $P^{(k)} = P(t)Q^k$, so the result is immediate.

Lemma

A matrix Qon finite set $I \times I$ is a Q-matrix if and only if $P(t)$ is a transition matrix for each $t \geq 0$.

Proof:
First if $P(t)$ is a transition matrix for all $t > 0$, then it is so for all t small.
To first order in t

$$P_{ij}(t) = \delta_{ij} + q_{ij}t$$

so to first order $\forall i \sum_j P_{ij}(t) = 1 + t(\sum_j q_{ij})$ which implies that for each $j \neq i,\ q_{ij} \geq 0$ and that $\sum_j q_{ij} = 0$.

Conversely, suppose that Q is a q-matrix and P is the coresponding semigroup. We note that for any matrix A (on $I \times I$)

$$\sum_k (AQ)_{ik} = \sum_k \sum_j A_{ij} Q_{jk} = \sum_j A_{ij} \sum_k q_{jk} = 0$$

In particular we have for every strictly positive integer r, $\sum_k Q^r_{ik} = 0$. So

$$\sum_k P_{ik}(t) = \delta_{ik} + \sum_r \frac{t^r}{r!} \sum_k Q^r_{ik} = 1$$

To complete the proof we notice that this must imply that to order t $P(t)$ is a transition semigroup for t small. We then write $P(t) = P(t/n)^n$ for general t and n large.Q