Exercise Set 6 30 March, 2021

Exercise 1. (**Markov Chain in a library**) In a library with n books, the ith book has probability p_i to be chosen at each request. To make it quicker to find the book the next time, the librarian moves the book to the left end of the shelf. Define the state of a Markov chain at any time to be the list of books we see as we examine the shelf from left to right. Since all the books are distinct, the state space E is the set of all permutations of the set $\{1, 2, \dots, n\}$. Show that

$$\pi(i_1, \cdots i_n) = p_{i_1} \cdot \frac{p_{i_2}}{1 - p_{i_1}} \cdots \frac{p_{i_n}}{1 - p_{i_1} - \cdots p_{i_{n-1}}}$$

is a stationary distribution.

Exercise 2. (**Random walk on a graph**)
An undirected graph $\mathcal{G}$ is a countable collection of states (that we call vertices) along with some edges connecting them. The degree d_i of a vertex i is the number of edges incident to i. We suppose the graph to be locally finite (i.e., each edge is incident to a finite number of edges). We say that a Markov chain on the state space $E = \mathcal{G}$ is a random walk on the graph if the transition probabilities are given by

$$p_{i,j} = \begin{cases} 1/d_i & \text{if } (i,j) \text{ is an edge,} \\ 0 & \text{otherwise,} \end{cases}$$

for $i, j \in \mathcal{G}$.

a) We assume that $\mathcal{G}$ is connected (implying that P is irreducible) and that $\sum_i d_i < \infty$. Find the stationary distribution of the random walk on $\mathcal{G}$.
Hint: Assume that the random walk is reversible and find a stationary distribution verifying the detailed balance equations. Explain why P is reversible.

b) We assume now that the graph is a chessboard, i.e., the vertices are $\mathcal{G} = \{1, \dots, 8\}^2$ and the edges are the possible moves of a King. We assume that the King starts its random walk in one of the four corners of the chessboard $c \in \mathcal{G}$. Compute the mean return time to the initial state $\mathbb{E}_c(T_c)$ of the King. Compute the same quantity for a Knight instead of a King.

Exercise 3. Let P be a transition matrix on a finite state space E.

(a) Prove the following linear algebra result: Given a matrix Q, Q and Q^t have the same eigenvalues.
Use this to prove that P has a stationary distribution π(i.e. a probability measure $\pi P = \pi$).

(b) Find an example, when E is an infinite state space, for which P doesn't have any stationary distribution.

Exercise 4. Let X be a Markov chain on E (not necessarily irreducible). Suppose that state $j \in E$ is positive recurrent and aperiodic. Show that

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j \mathbb{P}(\tau_j < \infty \mid X_0 = i), \qquad \tau_j = \inf\{n \geq 0 : X_n = j\},$$

where π is the stationary distribution of the chain restricted to the communicating class of j.

Exercise 5. Let X be a Markov chain with transition matrix P on $E = \{1, 2, 3, 4, 5\}$ given by

$$P = \begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} & 0 & \frac{3}{4} \end{pmatrix}.$$

(a) Find the communicating classes of P. For the recurrent classes, find the corresponding stationary distributions.

(b) Supposing that $X_0 \sim \alpha$ for a distribution α on E, find the limiting distribution of X_n when $n \to \infty$.
Hint: Suppose that X starts in a transient state of E and find the limiting distribution in this case.

Exercise 6. Let $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ be two independent Markov chains, aperiodic and irreducible, defined on the state spaces E and E', respectively. Show that $(X_n, Y_n)_{n\geq 0}$ is an aperiodic and irreducible Markov chain on $E \times E'$. Find an example of $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ independent and irreducible, but for which $(X_n, Y_n)_{n\geq 0}$ is not irreducible.

Exercise 7. (**Branching process with immigration**) For $n \in \mathbb{N}$, let $(N_k^n)_{k\geq 0}$ be a sequence of independent random variables on $\mathbb{Z}^+$ with a common generating function $\phi(t) = E(t^{N_k^n})$. The branching process with immigration is defined as

$$X_n = N_1^n + \ldots + N_{X_{n-1}}^n + I_n, \qquad n \geq 0,$$

where $(I_n)_{n\geqslant 0}$ is a sequence of independent random variables with values in $\mathbb{Z}^+$ with a common generating function $\psi(t) = E(t^{I_n})$. Show that if $X_0 = 1$ then

$$E(t^{X_n}) = \phi^{(n)}(t) \prod_{k=0}^{n-1} \psi(\phi^{(k)}(t)).$$

In the case where the number of immigrants in each generation is a Poisson random variable of parameter λ and $P(N_k^n = 0) = 1 - p$, $P(N_k^n = 1) = p$, find the proportion of time in the long run for which the population is 0.

Exercise 8. (**Metropolis–Hastings algorithm**) Suppose that we have a distribution p (called target distribution) on a countable space E. Then, for each $x \in E$, let q_x be a distribution on E (called the proposal distribution) with $q_x(y) > 0$ whenever $q_y(x) > 0$, for all $y \in E$. The Metropolis–Hastings algorithm constructs a Markov chain $(X_n)_{n\geq 0}$ as follows:

(i). Let $X_0 = x_0 \in E$ be random fixed state.

(ii). For $X_n = x$, choose a candidate y according to the proposal distribution q_x. In other words, with probability $q_x(y)$, the state y is the candidate state to which we may jump at time $n + 1$. Once we have a candidate state, we will decide if we jump to it or stay at x in the following way: Let U be a uniform random variable on $[0, 1]$, the variable X_{n+1} is defined as

$$X_{n+1} = \begin{cases} y & \text{if } U \leq \min\left(\frac{p(y)q_y(x)}{p(x)q_x(y)}, 1\right) \\ x & \text{otherwise.} \end{cases}$$

Show that if $(X_n)_{n\geq 0}$ is irreducible and aperiodic, then it is a reversible chain with respect to its stationary distribution p.