Exercise 1. Consider the following transition matrix:

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

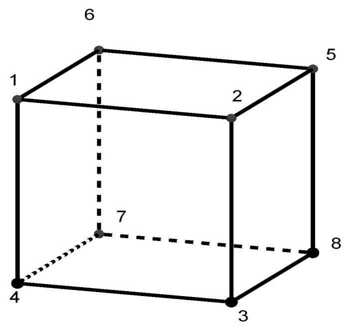
Determine which states are recurrent and which are transient.

Exercise 2. A particle moves on the eight vertices of a cube in the following way: at each step the particle is equally likely to move to each of the three adjacent vertices, independently of its past motion. Let the vertex 1 be the initial vertex occupied by the particle. Calculate each of the following quantities:

(a) the expected number of steps until the particle returns to 1,

(b) the expected number of visits to 8 until the first return to 1,

(c) the expected number of steps until the first visit to 8.



Exercise 3. (a) A transition matrix P defined on a state space E and a distribution λ have the *detailed balance property* if

$$\lambda_j P_{ji} = \lambda_i P_{ij},\ \forall i, j \in E.$$

Show that in this case, λ is a stationary distribution for P.

(b) Consider two urns each of which contains m balls; b of these $2m$ balls are black, and the remaining $2m - b$ are white. We say that the system is at state i if the first urn contains i black balls and $m - i$ white balls while the second contains $b - i$ black balls and $m - b + i$ white balls. Each trial consists of choosing a ball at random from each urn and exchanging the two. Let X_n be the state of the system after n exchanges have been made. X_n is a Markov chain.

(1) Compute its transition probability.

(2) Verify (using (a)) that the stationary distribution is given by

$$\pi(i) = \frac{\binom{b}{i}\binom{2m-b}{m-i}}{\binom{2m}{m}}.$$

(3) Can you give a simple intuitive explanation why the formula in (2) gives the right answer?

Exercise 4. Consider a Markov chain with state space $S = \{1, 2\}$ and transition matrix

$$\begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix},$$

$0 < a, b < 1$. Use the Markov property to show that

$$\mathbb{P}(X_{n+1} = 1) - \frac{b}{a+b} = (1 - a - b)\{\mathbb{P}(X_n = 1) - \frac{b}{a+b}\},$$

and conclude that

$$\mathbb{P}(X_n = 1) = \frac{b}{a+b} + (1 - a - b)^n\{\mathbb{P}(X_0 = 1) - \frac{b}{a+b}\}.$$

Further show that $\mathbb{P}(X_n = 1)$ converges exponentially fast to its limit distribution $b/(a + b)$.

Exercise 5. (**Reversible Processes**)

a) Let P be an irreducible matrix with stationary distribution π. We assume that $(X_n)_{0\le n\le N}$ is Markov(π, P). The process $Y_n = X_{N-n}$, $0 \le n \le N$ is called the *reverse process* of $(X_n)_{0\le n\le N}$. Show that $(Y_n)_{0\le n\le N}$ is Markov$(\pi, \hat{P})$, where $\hat{P} = (\hat{p}_{ij})$ is given by

$$\pi_j \hat{p}_{ji} = \pi_i p_{ij}, \ \forall\ i, j,$$

and $\hat{P}$ is also irreducible with stationary distribution π.

b) A transition matrix P is said to be *doubly stochastic* if its columns sum also to 1, that is $\sum_i p_{ij} = 1$ for all j.
Show that the stationary distribution of an irreducible Markov chain on N states is the uniform distribution ($\pi(i) = \frac{1}{N}$, $1 \le i \le N$) if and only if its transition matrix is doubly stochastic.

c) We say that an irreducible Markov chain $X \sim$ Markov(λ, P) is *reversible* if $\hat{P} = P$ (in that case λ should be stationary). Find an irreducible chain on $E = \{1, 2, 3\}$ with a stationary distribution but not reversible.

Exercise 6. Consider two boxes filled with gas molecules and joined by a small gap allowing them to pass from one box to the other. Assume that in total N molecules are in this configuration. We model the system so that at each time only one (randomly chosen) molecule is able to move from one box to the other.

(1) Show that the number of molecules in a box evolves according to a Markov process.

(2) Give the transition probabilities.

(3) What is the stationary distribution (detailed balance equations)?

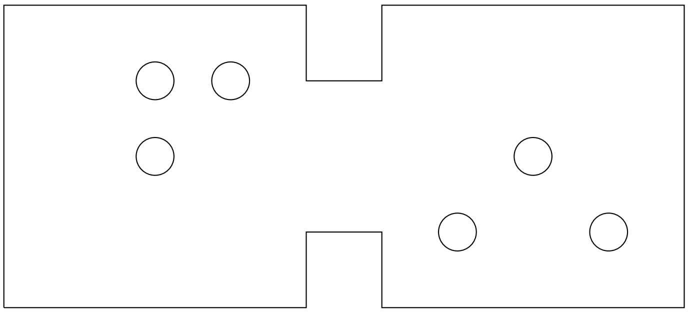


Figure 1: Configuration of the problem.

Exercise 7. Consider the aging chain on $\{0, 1, 2, \cdots\}$ in which for any $n \geq 0$ the individual gets one day older from n to $n+1$ with probability p_n but dies and returns to age 0 with probability $1 - p_n$. Find conditions that guarantee that

(a) 0 is recurrent,

(b) 0 is positive recurrent.

(c) Find the stationary distribution of the chain.