Exercise Set 4 March 15, 2021

Exercise 1 (Random walk). Let $(X_n)_{n\geq 0}$ be a one-dimensional random walk on the state space $\mathbb{Z}$ defined by the following transition probabilities:

$$P_{xy} = \begin{cases} p, & y = x+1, \\ q, & y = x-1, \end{cases}$$

(1) Prove that the random walk is recurrent if an only if $p = q$.

Hint: Note that $p_{00}^{2n+1} = 0$ for all $n \in \mathbb{N}$, and find the probability p_{00}^{2n}. You can then use Stirling's approximation to $n!$

$$n! \sim \sqrt{2\pi n}(n/e)^n, \quad n \to \infty.$$

(2) In the transient case $p \neq q$, find the limit $\lim_{n\to\infty} X_n$.

Exercise 2 (Birth and Death chain). Let us consider a Markov chain $(X_n)_{n\geq 0}$ on the state space $\mathbb{N}$ defined by the following transition probabilities:

$$P_{xy} = \begin{cases} p & \text{if } x > 0,\ y = x+1, \\ q & \text{if } x > 0,\ y = x-1, \\ 1 & \text{if } x = 0,\ y = 1. \end{cases}$$

Prove that:

(1) If $p \leq q$ the chain is recurrent.

Hint: study the probability $u(k) = \mathbb{P}_k\left(X_n \neq 0, \forall n \in \mathbb{N}\right)$ by showing that

$$u(k+1) - u(k) = \frac{q}{p}\left(u(k) - u(k-1)\right).$$

(2) If $q < p$ the chain is transient.

Hint: consider writing the chain as $X_n = \sum_{i=1}^{n} Y_i \mathbb{1}(X_{i-1} > 0) + |Y_i| \mathbb{1}(X_{i-1} = 0)$ where

$$Y_i \overset{\text{i.i.d.}}{\sim} \begin{cases} +1 \text{ with prob } p \\ -1 \text{ with prob } q \end{cases},$$

and compare with the non-symmetric random walk $\sum_{i=1}^{n} Y_i$.

Exercise 3. Let $Y_1, Y_2, \cdots$ be independent identically distributed random variables with $\mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = -1) = \frac{1}{2}$ and set $X_0 = 1$ and $X_n = X_0 + Y_1 + \cdots + Y_n$ for $n \geq 1$. Define the stopping time

$$H_0 = \inf\{n \geq 0 \mid X_n = 0\}.$$

(a) Find the probability generating function $\phi(s) = \mathbb{E}[s^{H_0}]$.

(b) Suppose that the distribution of the Y_i's is changed to $\mathbb{P}(Y_1 = 2) = \mathbb{P}(Y_1 = -1) = \frac{1}{2}$. Show that ϕ now satisfies

$$s\phi(s)^3 - 2\phi(s) + s = 0.$$

Exercise 4. (**Gambler's ruin**) Assume that a gambler is making bets for 1 dollar on fair coin flips, and that she will abandon the game when her fortune falls to 0 or reaches n dollar. Let X_t be the Markov chain on $\{0, \dots, n\}$ describing the gambler's fortune at time t, that is, $\mathbb{P}(X_{t+1} = k+1 \mid X_t = k) = \mathbb{P}(X_{t+1} = k-1 \mid X_t = k) = 1/2$, $k = 1, \cdots, n-1$, and $\mathbb{P}(X_{t+1} = 0 \mid X_t = 0) = \mathbb{P}(X_{t+1} = n \mid X_t = n) = 1$. Let T be the time required to be absorbed at one of 0 or n. Assume that $X_0 = k$, where $0 \leq k \leq n$.

(i). Find the probability $\mathbb{P}_k(X_T = n)$ for the gambler to reach n dollars with initial capital k.

(ii). Compute $\mathbb{E}_k[T]$, the expected time to reach n or 0 starting from k.
Hint: Writing $f_k := \mathbb{E}_k[T]$ and $\Delta_k = f_k - f_{k-1}$, show that $\Delta_k = \Delta_{k+1} + 2$ and use this to compute f_k.