Exercise 1. Consider the following transition matrix:

$$\begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{pmatrix}$$

(1) Identify the communicating classes of this matrix.

(2) What are the closed classes?

Exercise 2. Suppose that the probability it rains today is 0.3 if neither of the last two days was rainy, but 0.6 if at least one of the last two days was rainy. Let the weather on day n, W_n, be R for rain, or S for sun. W_n is not a Markov chain, but the weather for the last two days $X_n = (W_{n-1}, W_n)$ is a Markov chain with four states $\{RR, RS, SR, SS\}$.

(a) Find the transition matrix corresponding to X_n.

(b) What is the probability it will rain on Wednesday given that it did not rain on Sunday and Monday?

Exercise 3. A frog is jumping on the endpoints of a segment $[A, B]$ such that $p_{A,B} = \alpha$ and $p_{B,A} = \beta$, and the transition matrix is given by

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}.$$

Starting from point A, what is the probability that it will return to the same point after n jumps? In general, find the matrix P^n. Give conditions for the convergence $\lim_{n\to\infty} P^n =: P^\infty$ and compute this limit. How do you interpret the entries of the matrix P^∞.
Hint: Diagonalize the transition matrix. The entries of P^n are linear combinations of λ_1^n and λ_2^n, where λ_1 and λ_2 are the eigenvalues of P.

Exercise 4. Show that any finite transition matrix has at least one closed communicating class. Find an example of a transition matrix without any closed communicating class.

Exercise 5. Let $(X_n)_{n\geq 0}$ be a Markov chain on a state space E and a transition matrix P. For each state $i \in E$, we write τ_i for the duration of a visit in i (if $X_0 = i$, τ_i is then the moment when the Markov chain leaves i for the first time).
Determine the law of τ_i .