Exercise Set 11 May 19, 2021

Exercise 1. In each of the following cases, compute $\lim_{t\to\infty} \mathbb{P}(X_t = 2 \mid X_0 = 1)$ for the Markov chain $(X_t)_{t\geq 0}$ with the given Q-matrix on $\{1, 2, 3, 4\}$:

$$(a) \begin{pmatrix} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \qquad (b) \begin{pmatrix} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(c) \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 2 & -2 \end{pmatrix} \qquad (d) \begin{pmatrix} -2 & 1 & 0 & 1 \\ 0 & -2 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Exercise 2. Customers arrive at a single-server queue in a Poisson stream of rate λ. Each customer has a service requirement distributed as the sum of two independent exponential random variables of parameter μ. Service requirements are independent of one another and of the arrival process.

(a) Write down the generator matrix Q of a continuous-time Markov chain which models this, explaining what the states of the chain represent.

(b) Verify that the stationary distribution at state n is of the form $\pi_n = ax_1^n + bx_2^n$ with specific x_1 and x_2 and deduce that the chain is positive recurrent if and only if $\lambda/\mu < \frac{1}{2}$.

Exercise 3. Let $\{X(t) \mid t \in \mathbb{R}^+\}$ be a Markov process with n states $\{1, 2, \cdots, n\}$ and generator:

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda_2 & \lambda_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\lambda_3 & \lambda_3 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -\lambda_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\lambda_{n-1} & \lambda_{n-1} \\ \lambda_n & 0 & 0 & 0 & \cdots & 0 & -\lambda_n \end{pmatrix}.$$

(i). Find the stationary distribution of the process

(ii). Give an intuitive explanation of the result.

Exercise 4. Let $\{X(t) \mid t \in \mathbb{R}^+\}$ be a Markov process with n states $\{1, 2, \cdots, n\}$ and generator:

$$
Q = \begin{pmatrix}
-\lambda_1 & \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\
\lambda_2\mu_2 & -\lambda_2 & \lambda_2(1-\mu_2) & 0 & \cdots & 0 & 0 \\
\lambda_3\mu_3 & 0 & -\lambda_3 & \lambda_3(1-\mu_3) & \cdots & 0 & 0 \\
\lambda_4\mu_4 & 0 & 0 & -\lambda_4 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_{n-1}\mu_{n-1} & 0 & 0 & 0 & \cdots & -\lambda_{n-1} & \lambda_{n-1}(1-\mu_{n-1}) \\
\lambda_n & 0 & 0 & 0 & \cdots & 0 & -\lambda_n
\end{pmatrix}.
$$

(i). Guess the expression $\pi_1 = \lim_{t\to\infty} \mathbb{P}(X(t) = 1)$.

(ii). Find the stationary distribution of the process.

Exercise 5. Let $\{X(t) \mid t \in \mathbb{R}^+\}$ be an irreducible Markov process on a finite space of n states, with generator Q. Let us take λ so that $\lambda > \max_i\{-Q_{ii}\}$ and define the matrix:

$$
P = I + \frac{1}{\lambda}Q \qquad \text{(where } I \text{ is the identity matrix).}
$$

(i). Show that P is a transition matrix, and that its stationary distribution is identical to the one of Q.

(ii). Let $\{N(t) \mid t \in \mathbb{R}^+\}$ be a Poisson process with parameter λ and let $\{Y_k \mid k \in \mathbb{N}\}$ be a Markov chain with matrix P, independent from $\{N(t)\}$. Let $T_0 = 0$, and $T_1, T_2, \cdots$ denote the arrival times in the Poisson process.

We define the process $\{Z(t) \mid t \in \mathbb{R}^+\}$ as follows:

$$
Z(t) = Y_n \qquad \forall t \in [T_n, T_{n+1}[.
$$

Show that $\{Z(t)\}$ is a Markov process with generator Q.

Exercise 6. Show that the renewal function $R(t)$ satisfy a renewal equation, and specify the corresponding function $g(t)$.

Exercise 7. Let $S_1, S_2, \ldots$ be the successive times at which cars cross a certain fixed position on the highway. We assume that the intervals of time $W_1, W_2, \ldots$ between each renewal are i.i.d. with cumulative distribution $F(\cdot)$. Suppose that at time $t = 0$, a pedestrian arrives at this fixed position, and wants to cross the road. Assume that he needs τ units of time to cross it. Let L be the time that the pedestrian has to wait before starting to cross the road.

(a) Find the distribution of L and its expectation.

(b) Same questions if we assume that the arrivals of cars follow a Poisson process with parameter λ.