PRACTICE EXAM

Spring Semester

17 April 2023

Length of the exam : 1h45 (from13h15 to 15h00)

Attempt all the questions

First write your name, given names and section :

Name :______________________________ **Given Name :** ______________________________

Section : ____________________

Exercice	**Points**
1	
2	
3	
Total points :	

1)

a) Consider the Markov chain on $I = \{0, 1, 2\}$, where

$$\forall i \; P_{ii+1} = P_{ii} = 1/2$$

where $2 + 1$ is taken to be 0. We say that a clockwise circuit occurs at time n if $X_n = 0$, $X_{n+1} = 1, X_{n+2} = 2$ and $X_{n+3} = 0$. What is the expected number of circuits that occur in time interval $[0, 10000]$ approximately ?Justify.

b) A discrete time Markov chain on $I = \{1, 2, 3, 4\}$ has invariant distribution $(1/2,\ 1/8,\ 1/8,\ 1/4)$. Making an additional assumption, give the expected number of visits to state 3 by the Markov chain starting at 1, before returning to 1

c) Give the definition (not an example!) of a stopping time for a Markov chain $(X_n)_{n\geq 0}$. IGive an example of a positive integer valued random variable dderived from a MC which is not a stopping time.

d) For a discrete time Markov chain, if state i leads to state j and i is recurrant, must j lead i ? Justify or provide a common example.

e) For a (discrete time) Markov chain on $\{0, 1\}$ with transition matrix

$$P = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix}$$

find P^n_{01} for general n .

f) Give an example of a Markov chain which is not irreducible but which has a unique stationary distribution.

g) For transition matrix P, $P_{ij}P_{ji} > 0$. Is it necessarily true that the period of i is equal to that of j ?

h) Let λ be a probability on $I = \{1, 2 \cdots N\}$. Let transition probability P satisfy $P_{ij} = \lambda_j \; \forall i, j$. Give a necessary and sufficient condition for the chain to be irreducible. With this condition is it aperiodic ? Reversible with respect to λ ?

2)

Consider a discrete time Markov chain with transition matrix on $I = \{1, 2, 3, 4, 5\}$

$$P = \begin{pmatrix} 1/2 & 0 & 1/4 & 1/4 & 0 \\ 1/4 & 0 & 1/2 & 0 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

a) Find the probability of reaching 4 starting from 2 for the Markov chain.

b) Let $h(x) = E(\sum_{n=0}^{\infty} I_{X_n=1} \mid X_0 = x)$
Calculate $h(2)$

c) For an irreducible Markov chain on (finite or countable) state space I, let i and j be distinct sites and define
$h(x) = \quad E\left(\sum_{n=0}^{T_j} I_{X_n=i} \mid X_0 = x\right)$

where $T_j = \inf\{n \geq 0 : \quad X_n = j\}$

(i) Show $h(x) \; < \infty \quad \forall x \in I$

(ii) Give a characterization (via a system of equations) for h without proof.

3)

Consider a Markov chain X on $\mathcal{N} = \{1, 2, 3 \cdots\}$
with $P_{i\ i+1} = \left(\frac{i}{i+1}\right)^2 \quad P_{i\ 1} = 1 - \left(\frac{i}{i+1}\right)^2$

a) Show the chain is irreducible. Is it periodic ?

b) Show the chain is positive recurrant and give the stationary distribution π. Is the chain reversible for π ?

c) What can you say about the number of times that the chain has visited 3 by time 10^8 if $X_1 = 1$.