FINAL EXAM

JUNE 19 2019

Length of the exam : 3h00 (from12h15 to 15h15)

Attempt all the questions

First write your name, given names and section :

Name :______________________________ **Given Name :** __________________________

Section : ______________________

Exercice	**Points**
1	
2	
3	
4	
5	
Total points :	

1i) Give an example of an irreducible continuous time Markov chain which is not positive recurrent but for which the jump chain is.

(ii) Consider an irreducible Markov chain $(X_n)_{n\geq 0}$ in equilibrium and such that

$$(X_0, X_1, \cdots X_N) \quad \stackrel{D}{=} \quad (X_N, X_{N-1}, \cdots X_0)$$

for any $N > 1$. Show that the equilibrium distribution satisfies the detailed balance equations for the transition probabilities.

(iii) Consider the Markov chain on $I \;=\; \{0, 1, 2\}$, where

$$\forall i \; P_{ii+1} \;=\; P_{ii} \;=\; 1/2$$

where $2+1$ is taken to be 0. We say that a clockwise circuit occurs at time n if $X_n = 0,\;\; X_{n+1} = 1, X_{n+2} = 2$ and $X_{n+3} = 0$. What is the expected number of circuits that occur in time interval $[0, 10000]$ approximately ?

(iv) Give the definition (not an example !) of a stopping time for a Markov chain $(X_n)_{n\geq 0}$. Is (for the preceding example)

$$T \;=\; \inf\{n \geq 0 : \text{ a clockwise circuit occurs at } n\}$$

a stopping time for the chain ?

(v) Given Q-matrix

$$Q \;=\; \begin{pmatrix} -2 & 1 & 1 \\ 2 & -4 & 2 \\ 2 & 1 & -3 \end{pmatrix}$$

Give the corresponding jump chain transition matrix.

(vi) Give two nontrivial criteria for a continuous time Markov chain to be nonexplosive.

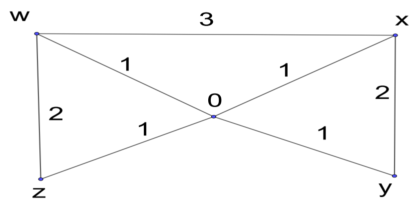
2) A woman possesses 3 umbrellas. Every morning (including weekends) she leaves her house and travels to her place of work and returns in the evening. For each journey (whether to or from home) she takes an umbrella if it is raining when she leaves and does not take an umbrella if it is not (even if she arrived with an umbrella !). For each journey it is raining with probability 1/2 independently of other journeys (including journeys on the same day). Formulate a Markov chain model to answer the following questions :
(i) approximately what proportion of journeys does she get wet ?
(ii) if she initially begins on day 0 at home with 3 umbrellas in her house, what is the chance that the first time she gets wet is while leaving home (and not leaving from the office) ?
(iii) What approximately is the chance that she gets wet on day 100 (remember there are two journeys for a single day).

3)People arrive at a taxi stand at rate λ and seat themselves in the taxi that is first in line. When this taxi has 4 people in it, it leaves immediately and is instantly replaced by an another taxi.
(i) Let $V(t)$ be the amount of time that the taxi that is first in line at time t spends between becoming first in line (at a time $< t$) and leaving (at a time $> t$). Give the asymptotic distribution of $V(t)$ as t becomes large.
(ii) Write down an exact formula for the probability that no taxi leaves in time $(0, 3)$ (assuming that at time 0 the leading taxi is empty.
(iii) If now leaving taxis are not immediately replaced but instead take exactly one unit of time to be replaced, give the asymptotic probability as t tends to infinity that a taxi is available at time t.
b) Let X_i be i.i.d. positive random variables and let Y_i also be i.i.d. and also independent of the X_i but not necessarily with the same distribution. Let the respective distribution functions be F and G. Define $A(t)$ to be the probability that for some $k, t \in [\sum_1^k (X_i + Y_i), \sum_1^k (X_i + Y_i) + X_{k+1})$. Explain a motivating example for the study of $A(.)$ and deduce the equation

$$A(t) \;=\; (1 - F(t)) + \int_0^t A(t-s) d(F \times G)(s).$$

Deduce that if $F \times G$ is nonarithmetic, $\lim_{t\to\infty} A(t) \;=\; \frac{\mu_X}{\mu_x + \mu_Y}$ where $\mu_X = E(X_1), \mu_Y = E(Y_1)$.

4) For the Markov process on the graph below only jumps between vertices connected by an edge are possible. The jump rates in either direction are the same and are indicated on the diagram.



(i)What is the probability that the chain starting at 0 hits site x before returning to 0 ?

(ii) What is the expected number of jumps made by the chain before hitting site x ?

(iii) Find a stationary probability for the process. Is it unique ?

5) Let $(X_n)_{n\geq 0}$ be a birth and death chain on $\{0,1,2,3,\cdots\}$ with $\forall i \geq 0$

$r_i = P(X_1 = i|X_0 = i)$, $p_i = P(X_1 = i+1|X_0 = i)$ and $q_i = P(X_1 = i-1|X_0 = i)$

(Obviously $q_0 = 0$.)
(i). Give a necessary and sufficient condition for the chain to be irreducible. With this condition give a necessary and sufficient condition for the chain to be aperiodic.
(ii) Derive the condition for the chain to be transient (under irreducibility)
(iii) For the transient case with $X_0 = 0$, what is the distribution of the number of visits of the chain to 0 ?