FINAL EXAM

Spring Semester 2021

July 2021

Length of the exam : 3h00 (from 8h15 to 11h15)

Attempt all the questions

First write your name, given names and section :

Name :_____________________________ **Given Name :** _____________________________

Section : ____________________

Exercice	**Points**
1	
2	
3	
4	
5	
Total points :	

1) FOR THIS QUESTION, I DO NOT EXPECT DETAILED IN DEPTH PROOFS BUT SHORT JUSTIFICATION SHOULD BE GIVEN

i)Let $P =$

$$P = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Is a Markov chain with this transition probability irreducible?

(ii) Consider a Markov chain $(X_n)_{n\geq 0}$ on $\mathbb{Z}_+$ with transition matrix, P, satisfying

$$P_{ij} = 0 \text{ for } |i-j| > 1.$$

Give necessary and sufficient conditions for the chain to be irreducible. Give necessary and sufficient conditions for it to be aperiodic in the case where the chain is irreducible.

(iii) Given a Markov chain $(X_n)_{n\geq 0}$, define a stopping time T.

(iv) True or False? Justify! A Markov chain on $I = \{1,2,3,4,5\}$ must have an invariant distribution.

(v) For transition matrix on $I = \{1,2,3,4\}$

$$P = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix}$$

What is $\mathbb{E}^1(T_4)$ where $T_4 = \inf\{n \geq 0 : X_n = 4\}$.

(vi) Customers arrive at a shop according to a Poisson process of rate 1/hour. Given that 3 customers arrived during time inteval $[2,5]$, what is the probability that exactly one customer arrived in each of the intervals $[2,3]$, $[3,4]$ and $[4,5]$?

(vii) For continuous time Markov chain $(X_t)_{t\geq 0}$ on $\{1,2,3,4\}$, the above matrix is the jump chain and for state i, the expected value of the time spent at i before jumpng is equal to i for $i = 1,2,3,4$. Give the Q-matrix for X.

(viii) Give an example of an irreducible continuous time Markov chain $(X_t)_{t\geq 0}$ which is not positive recurrent but for which the for the jump chain is positive recurrent.

(ix) Consider continuous time Markov chain $(X_t)_{t\geq 0}$ on $\mathbb{N} = \{1, 2, \cdots n, \cdots\}$ with $q_{i\ i+1} = (i+1)^3$, $q_{i\ 1} = (i+1)^2$ for $i \geq 2$ and all other $q_{i\ j} = 0$ for i, j distinct. Is the chain explosive? Justify.

2) I throw a fair die repeatedly. Let $Y_0, Y_1, Y_2, \cdots Y_n, \cdots$ be the numbers rolled (they are i.i.d. uniform on $\{1,2,3,4,5,6\}$). We define $(X_n)_{n\geq 0}$ by $X_n = \max\{Y_0, Y_1, Y_2, \cdots Y_n\}$ (so with probability one the chain is absorbed at state 6).

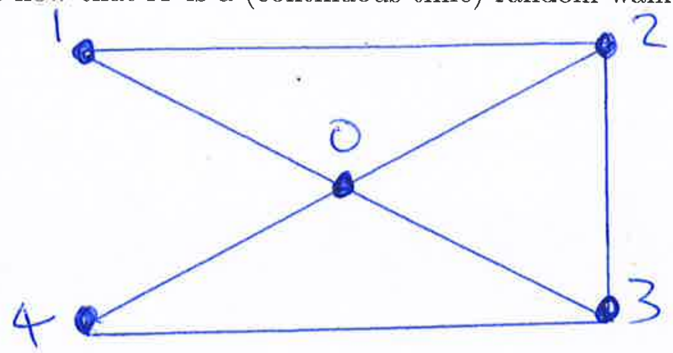
(i) $(X_n)_{n\geq 0}$ is a (λ, P) Markov chain. (You need not prove it is a Markov chain!). What is λ and P?

(ii) Is the chain aperiodic? Irreducible?

(iii) For $i \in \{1,2,3,4,5,6\}$, let function $h(i) = P(T_4 < \infty | X_0 = i)$. Find $h(i)$ for each i where as before. $T_4 = \inf\{n \geq 0 : X_n = 4\}$. Give corresponding linear equations for h. Is $P(T_4 < \infty | X_0 = i)$, the only solution for these equations?

(iv) Calculate the expected time the process spends in state 5.

3)What does it mean for a Markov chain $(X_n)_{n\geq 0}$, with transition matrix P, to satisfy the detailed balence equations for measure π on the state space?
(ii)For an irreducible continuous time chain $(X_t)_{t\geq 0}$, with $Q-$matrix Q, satisfying the detailed balance equations with measure π on the state space, show that the measure π is invariant for Q.
(iii) Give a criterion for X, as in (ii), to be positive recurrent (given that π is a distribution).
(iv) Suppose now that X is a (continuous time) random walk on the graph



so for the jump chain P_{ij} is zero if i and j are not neighbors and is otherwise $\frac{1}{d_i}$ where d_i is the degree of vertex i on the graph. The jump rate $q_i = 1$ for each state i. Find the equilibrium π. Can you generalize to general finite connected graphs satisfying these conditions? Is it true that for any X_0, $P(X_t = i) \to \pi(i)$? (Recall that a graph is connected if there exists a path between any two distinct sites.)

4)Customers arrive at a shop according to a Poisson process of rate λ. Each customer is a male with probability $\frac{1}{3}$ and a female with probability $\frac{2}{3}$ independently of other custiomers gender or arrival times. Each customer will buy something at the store with probability $\frac{1}{2}$ independently of all the previous information and of other customers purchasing decisions.

(i)What is the probability nothing is bought by a customer arriving in time $[0, 1]$?

(ii) If in $[0, 1]$ exactly 4 men arrived who made purchases, what is the conditional probability that during this time exactly 3 women came into this store.

(iii) Suppose now that on $[0, \infty)$ arriving women stay an exponential time with parameter c_F, while men stay exponential time with parameter c_H (again all these exponmential variables are independent of each other and arival times). Let F_t and H_t be the number of women (respectively men) in the shop at time t. For $V_t = (F_t, H_t)$ write down the Q matrix.

(iv) For which values of λ and c_F is the jump chain for F_t transient?

5)Consider the Markov chain on $S = \{0, 1 \cdots 6\}$ with transition matrix below

$$\begin{pmatrix} 0 & 1/3 & 0 & 1/3 & 0 & 0 & 1/3 \\ 1/2 & 0 & 1/6 & 0 & 1/6 & 1/6 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

a) What are the communicating classes?
b) What approximately is $P^{100}_{0\,3}$?
c) What approximately is $P^{100}_{0\,6}$?