Continuous time Markov Chains

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April 12, 2021

Why Exponential

We will study stochastic processes indexed by continuous time $t \geq 0, (X_t)_{\geq 0}$. We want that ror A an event depending on $X_s s \geq t$ that

$$\mathbb{P}(A|X_u \; 0 \leq u \leq t) = \mathbb{P}(A|X_t = i) = \mathbb{P}_i(A_t)$$

where A_t is the event A shifted leftwards by t temporally. That is if A is based on $X_{t+u} : \; u \geq 0$, then A_t is based on $X_u : \; u \geq 0$. Let us consider a Markov chain beginning at site i and let $A = \{X_u = i \;\; \leq u \leq t+s\}$. Let $\tau = \inf\{u \geq 0 \; X_u \neq i\}$. Then $\mathbb{P}_i(\tau \geq t+s|\tau \geq t) =$

$$\mathbb{P}_i(A|X_u = i \; 0 \leq u \leq t) = \mathbb{P}_i(A_t) = \mathbb{P}_i(\tau \geq s)$$

This gives

$$\mathbb{P}_i(\tau \geq t+s) = \mathbb{P}_i(\tau \geq t)\mathbb{P}_i(\tau \geq s)$$

This is the same as

$$log(\mathbb{P}_i(\tau \geq t+s)) = log(\mathbb{P}_i(\tau \geq t)) + log(\mathbb{P}_i(\tau \geq s)).$$

which leads to the existence of a constant (depending on i, q_i so that

$$\forall t > 0 \; log(\mathbb{P}_i(\tau \geq t)) = -q_i t$$

That is, under $\mathbb{P}_i, \tau$ is $\mathcal{E}xp(q_i)$. This is with slight abuse of notation: q_i could be both 0 or ∞.

Exponentials: A review.

If W is $\mathcal{E}xp(\lambda)$, then

- $E(W) = \frac{1}{\lambda}$
- density of W is $\lambda e^{-\lambda x}$ on $x > 0$.
- $P(W \in (t, t + dt | W > t) = \lambda dt + o(dt)$ Equally $W - t \stackrel{D}{=} W$ given $W > t$.

The last property is called the memoryless property

Theorem

Let I be a countable set and for $k \in I$ let S_k be independent $\mathcal{E}xp(q_k)$ with $\sum_j q_j < \infty$ random variables. If $S = \inf_k S_k$, then
(i) S is achieved by a single S_k
(ii) $P(S = S_k) = \frac{q_k}{q}$ where $q = \sum_j q_j$
(iii) S is $\mathcal{E}xp(q)$ and
(iv) r.v. K on I defined by $K = k$ iff $S = S_k$ is independent of S.

Proof

Let A_k be the even that $S_k > S_j \forall j \neq k$. We integrate over the density for S_k to obtain

$$P(A_k) = \int_0^\infty q_k e^{-q_k t} P(A_k | S_k = t) dt$$

$$= \int_0^\infty q_k e^{-q_k t} \prod_{j \neq k} e^{-q_j t} dt =$$

$$= \int_0^\infty q_k \prod_j e^{-q_j t} dt = \frac{q_k}{q}$$

Note that as $\sum_k \frac{q_k}{q} = 1$ all other possibilities have probability zero. It remains to give the distribution of S and show independence of K. This amounts to showing that $\forall t > 0$,

$$P(K = k, S > t) = P(K = k) P(S > t) = \frac{q_k}{q} P(S > t) = \frac{q_k}{q} e^{-qt}$$

But $S > t$ is exactly the event $S_j > t$ for each j which has probability

Explosions

Theorem

Let $S_k k \geq 1$ *be independent* $\mathcal{E}xp(\lambda_k)$ *r.v.s.*

If $\sum_k \frac{1}{\lambda_k} < \infty$, *then* $\sum S_k < \infty$

If $\sum_k \frac{1}{\lambda_k} = \infty$, *then* $\sum S_k = \infty$ *a.s*

Poisson processes

In the book a Poisson process is a Markov process on $\mathcal{N}$. It is derived from a given sequence of i.i.d. $\mathcal{E}xp(\lambda)$ random variables S_k. The parameter λ is called*the rate* of the process. Typically (but not always) the process starts from 0. If $X_0 = i$, then for all $t \geq 0$,

$$X_t = i + \sup\{k : S_1 + S_2 \cdots + S_k \leq t.\}$$

Note that a Poisson process starting at i is just a Poisson process starting at 0 with i added to it.

Theorem

For $(X_s)_{s\geq 0}$ and $t > 0$, given X_u $0 \leq u \leq t$,

$$(Y_s)_{s\geq 0} \equiv (X_{t+s})_{s\geq 0}$$

is a rate λ Poisson process starting at X_t.

Proof

We suppose (only to fix notation) that $X_0 = 0$ and that $X(t) = i$. Given $X_u 0 \leq u \leq t$, we know the values of $S_1, S_2, \cdots S_i$ and that $S_{i+1} > t$. The variables $S_k : k > i+1$ are independent of $X_u 0 \leq u \leq t$. Let is write for $k \geq 1$

- $\tilde{S}_k = S_{k+i}$ for $k > 1$
- $\tilde{S}_1 = S_{i+1} - t$

Then we have that conditional upon X_u $0 \leq u \leq t$ the variables $\tilde{S}$ are i.i.d. $\mathcal{E}xp(\lambda)$ and

$$X_{t+r} = i + \sup\{k : \tilde{S}_1 + \tilde{S}_2 \cdots + \tilde{S}_k \leq r$$

We will see that it is typically hard to calculate for a Markov chain $P_{ij}(t) \equiv \mathbb{P}(X_t = j | X_0 = i)$. However the Poisson process is special in that this is doable. We need only treat $i = 0$. The event that $X_t < j$ is simply the event that

$$S_1 + S_2 \cdots + S_j > t$$

for S_i i.i.d. $\mathcal{E}xp(\lambda)$. As is well known (and easily shown) the law of $S_1 + S_2 \cdots + S_j$ is a Gamma distribution with parameter j and λ. So r.v. $S_1 + S_2 \cdots + S_j$ has density

$$\frac{1}{(j-1)!} \lambda^j s^{j-1} e^{-\lambda s}$$

So $P(S_1 + S_2 \cdots + S_j > t) =$

$$\int_t^\infty \frac{1}{(j-1)!}\lambda^j s^{j-1} e^{-\lambda s} ds.$$

This via a succession of integration by parts becomes

$$e^{-\lambda t} + \lambda t e^{-\lambda t} \cdots + \frac{1}{(j-1)!}\lambda^{j-1} t^{j-1} e^{-\lambda t}$$

So $P_{0j}(t) = P(X(t) = j) = P(S_1 + S_2 \cdots + S_j + S_{J+1} > t)$ - $P(S_1 + S_2 \cdots + S_j > t) =$

$$\frac{\lambda^j t^j}{j!} e^{-\lambda t}$$

Equally if $X(0) = 0$, then $X(t)$ has the law of a Poisson(λt)

3 EQUIVALENT CONDITIONS

We give three equivalent conditions for a cadlag process $(X_t)_{t\geq 0}$ with initial value 0 on the positive integers to be a rate λ Poisson process.

Theorem

For $0 < \lambda < \infty$, the following are the same
(i) $(X_t)_{t\geq 0}$ is constructed via an .i.i.d sequence of $\mathcal{E}xp(\lambda)$ random variables, as above.
(ii) $(X_t)_{t\geq 0}$ has independent increments and as $h \to 0,\ P(X_{t+h} = X_t) = 1 - \lambda h + o(h)$ and $P(X_{t+h} = X_t + 1) = \lambda h + o(h)$.
(iii) $(X_t)_{t\geq 0}$ has stationary independent increments and the incremet of each interval of length t is a Poisson(λt).

Here independent increments means that

$$\forall n \forall 0 < t_1 < t_2 \cdots < t_n, (X_{t_i} - X_{t_{i-1}})$$

are independent. Stationary means that $\forall s, t > 0, X(t+s) - X(t) = X(s) - X(0)$ in distribution.

(i) is same as (iii)

In fact we already have $(i) \Rightarrow (iii)$ so it reamins to show that (iii) implies (i). This is immediate, modulo a measure theory fact for cadlag processes. Given property (iii), we know that for each n and each $0 < t_1 < t_2 \cdots < t_n$

$$P\left(X_{t_1} = i_1, X_{t_2} = i_2 \cdots X_{t_n} = i_n\right) = \prod_{j=1}^{n} P(Poisson(\lambda(t_j - t_{j-1}) = i_j - i_{j-1})$$

This implies that our (iii) process has the same finite dimensional distributions as (i). A basic result in measure theory for cadlag processes asserts that therefore the laws ogf (iii) and (i) are identical.

(ii) is same as (iii)

Firstly (iii) implies (ii): (iii) ensures independent increments and as h becomes small the probabilities for $X_{t+h} = X_t)$ or $X_{t+h} = X_t) + 1$ are simply properties of $Poisson(\lambda h)$ random variables. It reamins to show that (ii) implies that the increments are Poisson. Fix $t_{i-1} < t_i$ and divide up $[t_{i-1}, t_i]$ into n equal intervals of length $(t_i - t_{i-1})/n$. As n becomes large $h = (t_i - t_{i-1})/n$ becomes small. By independent increments, theevents $X(t_{i-1} + i(t_i - t_{i-1})/n) \neq X(t_{i-1} + (i-1)(t_i - t_{i-1})/n)i \geq 1$ are independent and have probability $\lambda h + o(h)$. So we have by usual probability arguement

$$\text{number } i : X(t_{i-1} + i(t_i - t_{i-1})/n) \neq X(t_{i-1} + (i-1)(t_i - t_{i-1})/n)$$

converges in distribution to a Poisson ($\lambda(t_i - t_{i-1})$). But the second condition implies that with probability tending to one as n tends to infinity, the above random variable is equal to $X(t_i) - X(t_{i-1})$. We are done.

Three Theorems

Theorem

For $0 < \lambda_1, \lambda_2 < \infty$, *let* $(X_i(t))_{t\geq 0}$ *be independent Poisson processes of rates* λ_1 *and* λ_2 *respectively. Then*

- $X(t) = X_1(t) + X_2(t)$ *is a rate* $\lambda_1 + \lambda_2$ *Poisson process*
- *Given* $X(u) 0 \leq u \leq t$ *(that is knowing* $X(t) - X(0) = n$ *and the jump times* $0 < t_1 < t_2 \cdots t_n < t$, *each jump time belongs to* X_i *with probability* $\frac{\lambda_i}{\lambda_1+\lambda_2}$ *independently of the others.*

Three Theorems

Theorem

Let $(X(t))_{t\geq 0}$ be a rate λ Poisson process and let $I_j : j \geq 1$ be i.i.d. Bernoulli (p) random variables independent of $X(.)$ If $0 < t_1 < t_2 \cdots$ are the jump times of X and Y, Z are constructed via X and the $I_j : j \geq 1$ by
$Y(t) = |\{j : t_j \leq t \text{ and } I_j = 1\}|$ and
$Z(t) = |\{j : t_j \leq t \text{ and } I_j = 0\}|$, then
Y and Z are independent Poisson processes of rates λp and $\lambda(1-p)$ respectively.

Three Theorems

Theorem

For $(X(t))_{t\geq 0}$a rate λ Poisson process starting at 0 , then given that $X(t) = n$, the jump times of X on $(0, t)$ are i.i.d. $U([0, 1])$ random variables ordered.