Countably infinite I

We consider the general case. Again the Markov chain is defined by a pair (λ, Q) where λ is just a probability vector on I giving the distribution of X_0 and Q is a Q-matrix:

- $q_{ij} = Q_{ij} \geq 0$ for $i \neq j$
- $\forall i \in I \sum_j q_{ij} = 0$ (or $-q_i \equiv q_{ii} = -\sum_{j \neq i} q_{ij}$

Again the cooresponding Markov chain stays at a site i for $\mathcal{E}xp(q_i)$ time before moving to site j with probability $\pi_{ij} = q_{ij}/q_i$ (if defined). Again we have a well defined jump chain $(Y_n)_{n \geq 0}$. So where is the problem?

Recall

Theorem

Let $S_k k \geq 1$ *be independent* $\mathcal{E}xp(\lambda_k)$ *r.v.s.*

If $\sum_k \frac{1}{\lambda_k} < \infty$, *then* $\sum S_k < \infty$

If $\sum_k \frac{1}{\lambda_k} = \infty$, *then* $\sum S_k = \infty a.s$

A pure birth process

For a pure birth process $I = \mathcal{N}$ and $\forall i q_i = q_{ii+1}$. That is the jump chain is trivial and deterministic: $\pi_{ii+1} = 1 \forall i$. The problem is that this still does not define a process $(X_t)_{t \geq 0}$.

If $S_i i = 1, 2 \cdots$ are the "holding times", then

$$\sum_i S_i < \infty a.s. \iff \sum_i 1/q_i < \infty.$$

If $\sum_i S_i < \infty a.s.$, then we say there is an *explosion* for the chain: X_t is not defined for $t \geq \sum_i S_i$.

General Explosions and the minimal chain

For a general (λ, Q) chain we can define the jump chain $(Y_n)_{n\geq 0}$ and given this we can choose $(S_n)_{n\geq 1}$ to be independent $\mathcal{E}xp(q_{Y_{n-1}})$ random variables. If

$$\sum_{n\geq 1} S_n < \infty$$

we say there is an explosion for the chain. In the case where an explosion occurs with strictly positive probability, we can still define $P_{ij}(t)$ by

$$P_{ij}(t) = P(X_t = j | X_0 = i)$$

but it is no longer true that $\sum_j P_{ij}(t) = 1 \forall i$.

Conditions for Nonexplosion

Theorem

There is a.s. no explosion for a (λ, Q) Markov chain if one of the following holds

- *I is finite*
- $\sup_i q_i < \infty$
- $\sum_{i\pi \text{ recurrent}} \lambda_i = 1$

Proof (of (iii): Note we do not require that π be irreducible. It is enough to show that for i recurrent for the discrete time jump chain $P(\sum_n 1/q_{Y_{n-1}} = \infty | Y_0 = i) = 1$.

But if i is recurrent with probability one $Y_n = i$ for infinitely many n (given $Y_0 = i$) and so necessarily a.s

$\sum_n 1/q_{Y_{n-1}} \geq 1/q_i + 1/q_i + \cdots + 1/q_i + \cdots = \infty.$

If we have a Markov chain for which expolosions occur with strictly positive probability one convention is to adjoin an absorbing point ∞ to I and to define $X_t = \infty$ if an explosion has occurred by time t. But it is important to realize that we could stipulate that at an explosion time e for the Markov chain, we stipulated that $X(e)$ was some point in I randomly chosen according to fixed but arbitrary law ν and from this point the chain evolved as a markov chain until the next explosion e_2 whereupon $X(e_2)$ was randomly chosen with probability ν and so on. The resulting process would be a Markov process.

Theorem

For a Q matrix, the equation

$$P'(t) = QP(t), \quad P(0) = I_d$$

has a minimal positive solution $P(t)$ which is given by $P(t)_{ij} = P(X_t = j | X_0 = i)$ where $(X_t)_{t \geq 0}$ is the process absorbed at ∞.

Remark: So if an explosion occurs with positive probability for some initial states i, then for some i, $\sum_{j \in I} P(t)_{ij} < 1$.

A second Theorem

We valso have

Theorem

The minimal semigroup for general Q-matrix Q is also the minimal positive solution to

$$P'(t) = P(t)Q., \quad P(0) = Id$$

We emphasize that if Q is exposive then there will exist many solutions to the forward and backward equations with initial conditions $P(0) = Id$.

An equivalence

Though there exist non minimal stochastic processes corresponding to a pair (λ, Q)which have the Markov property, in this course we will restrict attention to the minimal Markov chains. These will be what we consider to be continuous time markov chains. The following gives two equivalent ways of looking a these Markov chains.

Theorem

Given a general Q-matrix Q and the associated minimal semigroup $P(t)_{t\geq 0}$ the following are equivalent for a process $(X_t)_{t\geq 0}$ that is cadlag on I

a) Conditional upon $X_0 = i$, the jump chain of X, $(Y_n)_{n\geq}$ is a discrete time (δ_i, Π) Markov chain (where Π is derived from Q in the usual way) and conditional upon $(Y_n)_{n\geq}$, the holding times $(S_n)_{n\geq 1}$ are independent $\mathcal{E}xp(q_{Y_{n-1}})$ random variables.

b) Given integer $n \geq 0$ and times $0 \leq t_0 < t_1 < \cdots < t_{n+1}$ and $i_0, i_1 \cdots i_{n+1} \in I$,

$$P(X_{t_{n+1}} = i_{n+1} | X_{t_0} = i_0, X_{t_1} = i_1 \cdots X_{t_n} = i_n) = P_{i_n i_{n+1}}(t_{n+1} - t_n)$$

Definition:

For a Markov chain X a stopping time is a random variable taking values in $[0, \infty]$ so that for all $t \geq 0$, the event $T \leq t$ is equal to $X|_{[0,t]} \in A$ for some Borel subset A of the space of cadlag paths from $[0, t]$ to I

Theorem

For X a (λ, Q) Markov chain with stopping time T, conditional upon $T < \infty$ and $X_T = i$, the process

$$(Y_s)_{s\geq 0} \qquad Y_s \equiv X_{T+s}$$

is a (δ_i, Q) Markov chain independent of $X_s, s \in [0, T]$