Exercise Set 7

April 21, 2021

Exercise 1. A fair die is thrown repeatedly. Let X_n denote the sum of the first n throws. Find

$$\lim_{n\to\infty} \mathbb{P}(X_n \text{ is a multiple of } 13).$$

Exercise 2. In each of the following cases determine whether the stochastic matrix P, which you may assume is irreducible, is reversible:

(a)

$$\begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

(b)

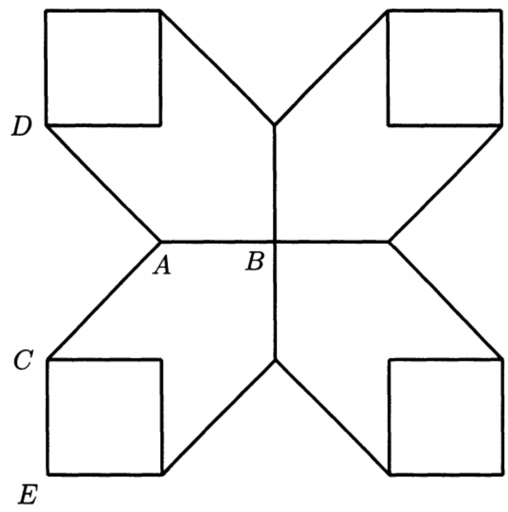
$$\begin{pmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{pmatrix}$$

(c) $I = \{0, 1, \cdots, N\}$ and $p_{ij} = 0$ if $|j - i| \geq 2$

(d) $I = \{0, 1, 2, \cdots\}$ and $p_{01} = 1, p_{i,i+1} = p, p_{i,i-1} = 1 - p$ for $i \geq 1$

(e) $p_{ij} = p_{ji}$ for all i, j in the state space S.

Exercise 3. Two particles X and Y perform independent random walks on the graph shown in the diagram. So for example, a particle at A jumps to B,C or D with equal probability $\frac{1}{3}$.



Find the probability that X and Y ever meet at a vertex in the following cases:

(a) X starts at A and Y starts at B

(b) X starts at A and Y starts at E.

For $I = B, D$, let M_I denote the expected time, when both X and Y start at I, until they are once again both at I. Show that $9M_D = 16M_B$.

Exercise 4. A professor has N umbrellas. He walks to the office in the morning and walks home in the evening. If it is raining he likes to carry an umbrella and if it is fine he does not. Suppose that it rains on each journey with probability p, independently of past weather. What is the long-run proportion of journeys on which the professor gets wet?

Exercise 5 (Countable exponential races). Let I be a countable space and let $T_k, k \in I$, be independent exponential random variables with $T_k \sim Exp(q_k)$ with $0 < q := \sum_{k \in I} q_k < \infty$. Set $T = inf_k T_k$. Let K be the random variable with values in I that is equal to k whenever $T = T_k$ and $T_j > T_k$ for $j \neq k$. Show that T and K are independent with $T \sim Exp(q)$ and $\mathbb{P}(K = k) = q_k/q$. Deduce that $\mathbb{P}(K = k \text{ for some } k) = 1$.

Exercise 6 (General construction of Markov processes). Let us consider a countable state space E and an array of positive numbers $(\lambda_{i,j})_{i,j \in E; i \neq j}$ with $\sum_{j \in E; j \neq i} \lambda_{i,j} < \infty$ for all $i \in E$. We recursively define a continuous time stochastic process $(X(t))_{t \geq 0}$ on E starting at $i_0 \in E$ as follows:

(i). Define $T_0 = 0$ and set $X(T_0) = i_0 \in E$;

(ii). For $n \in \mathbb{N}$: suppose we know T_{n-1} and $X(T_{n-1}) = i_{n-1}$. Independently of the previous steps, generate independent exponential random variables $E_1, E_2, \dots$ with $E_j \sim Exp(\lambda_{i_{n-1},j})$. Define $T_n = T_{n-1} + \inf_{j \in \mathbb{N}} E_j$ and $i_n = \text{argmin}_{j \in E} E_j$, that is, the (random) index of the exponential variable that is the smallest. Then put

$$X(t) = \begin{cases} i_{n-1} & \text{for } t \in [T_{n-1}, T_n) \\ i_n & \text{for } t = T_n. \end{cases}$$

a) What is the distribution of the time between the jumps of the process $(X(t))_{t \geq 0}$?

b) Let $\widehat{P}_{ij}$ be the probability

$$\widehat{P}_{ij} = \mathbb{P}(X(T_n) = j \mid X(T_{n-1} = i)).$$

Find the matrix $\widehat{P} = (\widehat{P}_{ij})_{i,j \in E}$.

c) Show that $(X(t))_{t \geq 0}$ is a homogeneous Markov process.

Definition (The Q-matrix).
One way of thinking about the evolution of the Markov process $(X(t))_{t \geq 0}$ is in terms of its Q-matrix, which is known as the generator of the process. A matrix $Q = (q_{ij})_{i,j \in E}$ is a Q-matrix if it satisfies

(i). $-\infty < q_{ii} \leq 0$ for all $i \in E$;

(ii). $0 \le q_{ij} < \infty$ for all $i \neq j$;

(iii). $\sum_{j \in E} q_{ij} = 0$ for all $i \in E$.

The Q-matrix of the Markov process $(X(t))_{t \ge 0}$ as constructed above is given by $q_{ii} = -\sum_{j \neq i} \lambda_{i,j}$ for $i \in E$, and $q_{ij} = \lambda_{ij}$ for $j \neq i$.

Exercise 7. In a population of size N, a rumor is begun by a single individual who tells it to everyone he meets; they in turn pass the rumor to everyone they meet, once a person has passed the rumor to somebody he exits the system. Assume that each individual meets another randomly with exponential rate $1/N$. Let $X(t)$, $t \ge 0$ be the number in $E = \{1, \ldots, N\}$ of people who know the rumor at time t.

a) Draw a graph to visualize the chain. Write down the Q-matrix of the chain.

b) How long does it take in average until everyone knows the rumor if $X(0) = 1$?

Exercise 8 (Poisson process). For $i \in \mathbb{N}$, let E_i be independent copies of an exponential random variable of parameter λ. We let $T_n := E_1 + \cdots + E_n$ and

$$N(t) := \sum_{n=1}^{\infty} \mathbb{1}_{\{T_n \le t\}}, \; t \ge 0.$$

The process $(N(t))_{t \ge 0}$ is called a homogeneous Poisson process with intensity λ. Let $T_0 = 0$ and we say that $T_1, T_2, T_3, \ldots$ are the successive arrival times of the Poisson process, and E_n the intervals $T_n - T_{n-1}$.

(i). Show that T_n follows an Erlang law with parameters n and λ having density:

$$f_{T_n}(t) = \frac{\lambda^n}{(n-1)!} \, t^{n-1} \, e^{-\lambda t} \, \mathbb{1}_{\{t > 0\}}.$$

(ii). Show that, $\forall t > 0$, $N(t)$ follows a Poisson law with parameter λt, i.e.

$$\mathbb{P}(N(t) = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \quad k = 0, 1, 2, \ldots.$$