Exercise 1. A fair die is thrown repeatedly. Let X_n denote the sum of the first n throws. Find

$$\lim_{n\to\infty} \mathbb{P}(X_n \text{ is a multiple of } 13).$$

Solution. Let $[i]$ be the set of numbers for the which the remainder of their division by 13 is equal to i, $i = 0, \cdots 12$. If $X_n \in [i]$, X_n has equal probabilities to belong to $[i+1], \cdots, [i+6]$ where $[13] = [0], [14] = [1], \cdots$. So the transition matrix corresponding to X_n and the states $[0], \cdots [12]$ is given by

$$P = \begin{bmatrix}
0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

This matrix is doubly stochastic and so it has a uniform stationary distribution $\pi = (\frac{1}{13}, \cdots, \frac{1}{13})$. So we get

$$\lim_{n\to\infty} \mathbb{P}(X_n \text{ is a multiple of } 13) = \lim_{n\to\infty} \mathbb{P}(X_n \in [0]) = \pi_{[0]} = \frac{1}{13}.$$

Exercise 2. In each of the following cases determine whether the stochastic matrix P, which you may assume is irreducible, is reversible:

(a)

$$\begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

(b)

$$\begin{pmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{pmatrix}$$

(c) $I = \{0, 1, \cdots, N\}$ and $p_{ij} = 0$ if $|j - i| \geq 2$

(d) $I = \{0, 1, 2, \cdots\}$ and $p_{01} = 1, p_{i,i+1} = p, p_{i,i-1} = 1 - p$ for $i \geq 1$

(e) $p_{ij} = p_{ji}$ for all i, j in the state space S.

Solution. (a) Since the matrix is irreducible, we have that $p, q > 0$. The chain is reversible if and only if it has a stationary distribution $\pi = (\pi_1, \pi_2)$ verifying the detailed balance equations $\pi_1 P_{12} = \pi_2 P_{21}$. Since $\pi_1 + \pi_2 = 1$, we get the solution $\pi_1 = \frac{1}{1+\frac{p}{q}}$ and $\pi_2 = \frac{p/q}{1+p/q}$.

(b) We need to find π verifying

$$\pi_1 p = \pi_2(1-p), \quad \pi_1(1-p) = \pi_3 p, \quad \pi_2 p = \pi_3(1-p), \ \pi_1 + \pi_2 + \pi_3 = 1.$$

We have a solution for this system of equations given by

$$\pi_1 = \frac{1}{1 + p/(1-p) + (1-p) + p^2/(1-p)}, \quad \pi_2 = \frac{p/(1-p)}{1 + p/(1-p) + (1-p) + p^2/(1-p)},$$
$$\pi_3 = \frac{1 - p + p^2/(1-p)}{1 + p/(1-p) + (1-p) + p^2/(1-p)},$$

and so the chain is reversible whenever $p = 1/2$.

(c) Write for any $i \leq N$ $q_i = P_{i,i-1}, \ r_i = P_{i,i}, \ p_i = P_{i,i+1}$. The chain is again reversible since we can find a solution for the system of equations

$$\pi_i p_i = \pi_{i+1} q_{i+1} \Longrightarrow \pi_{i+1} = \pi_0 \frac{\prod_{j=0}^{i} p_j}{\pi_{j=1}^{i+1} q_j}.$$

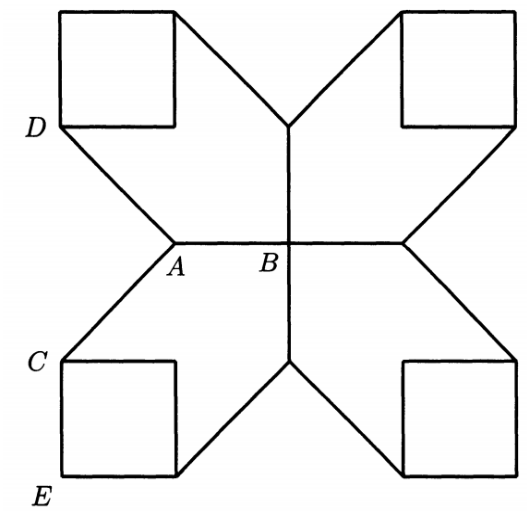
(d) The detailed balance equations in this case are given by

$$\pi_0 = (1-p)\pi_1, \quad \pi_i p = (1-p)\pi_{i+1}, \ i \geq 1.$$

From this we get $\pi_i = \left(\frac{p}{1-p}\right)^i$. We need that $\sum_j \pi_j = 1$ so we need that $\pi_i \to 0$ as $i \to \infty$. We deduce that we have a solution for the system (and thus the chain is reversible) as long as $p < 1 - p$ which is equivalent to $p < \frac{1}{2}$.

(e) In this case the matrix is doubly stochastic and so it has a uniform stationary distribution ($\pi_i = \pi_j \ \forall i, j \in S$). The detailed balance equations are clearly satisfied in this case and so the chain is reversible.

Exercise 3. Two particles X and Y perform independent random walks on the graph shown in the diagram. So for example, a particle at A jumps to B,C or D with equal probability $\frac{1}{3}$.



Find the probability that X and Y ever meet at a vertex in the following cases:

(a) X starts at A and Y starts at B

(b) X starts at A and Y starts at E.

For $I = B, D$, let M_I denote the expected time, when both X and Y start at I, until they are once again both at I. Show that $9M_D = 16M_B$.

Solution. (a) From the graph, it is easy to see that both A and B have period 2, that means, starting from A (resp. B), we can only return to A (resp. B) in an even number of steps. Suppose that X and Y meet at C without loss of generality, after k steps. Then there's a path of length $k+1$ starting from A and returning to A, and a path of length $k+2$ starting from B and returning to B. Since $k+1$ and $k+2$ cannot be both even numbers, this is a contradiction with the 2-periodicity of A and B. We deduce that X and Y will never meet in this case and so the probability of them meeting is 0.

(b) Writing G for the graph in the picture, we let $G' := G \times G$ having vertices $v_1 \times v_2$ for any vertices $v_1, v_2 \in G$ and (u_1, u_2) is connected to (v_1, v_2) if there's an edge between u_1 and v_1 as well as between u_2 and v_2 in G.
So we can see the random walk as one particle (X, Y) moving on the graph G'. The communication class of (A, E) is irreducible (by definition) and finite, hence the random walk will reach all the states this class with probability 1. In particular, (C, C) belongs to this class and so with probability 1 the two particles will meet at point C.

As seen in the previous exercise sheet, the stationary distribution of the random walk on the graph is of the form $\pi_i = \frac{d_i}{\sum_j d_j}$ for a vertex i in the graph where d_i indicates the degree (number of neighbors) of i. In our case, (D, D) has degree 9 (since D has degree 3) and (B, B) has degree 16 since B has degree 4.
Write G'' for the subgraph consisting of the communication class of (D, D) (that contains also (B, B)). So starting from (D, D), we need on average

$$M_D = \frac{1}{\pi_{(D,D)}} = \frac{\sum_{j \in G''} d_j}{9}$$

to return to (D, D). Similarly, the expected time to return to (B, B) starting from (B, B) is given by

$$M_B = \frac{1}{\pi_{(B,B)}} = \frac{\sum_{j \in G''} d_j}{16}.$$

So we deduce that $9M_D = 16M_B$.

Exercise 4. A professor has N umbrellas. He walks to the office in the morning and walks home in the evening. If it is raining he likes to carry an umbrella and if it is fine he does not. Suppose that it rains on each journey with probability p, independently of past weather. What is the long-run proportion of journeys on which the professor gets wet?

Solution. We start by writing the transition matrix corresponding to the number of umbrellas at home after one day (2 journeys).
If he has 0 umbrellas at home, the probability that it stays 0 after 1 day is the probability that it doesn't rain on his way home from work, that is $1-p$, otherwise he wil have 1 umbrella at home at the end of the day (with probability p).
Fix $i \in \{1, \cdots, N-1\}$ and suppose that he has i umbrellas at home at the begining of the day. The probability that the number of umbrellas becomes $i-1$ at the end of the day is the probabiity that it rains on his way to the office and it doesn't rain on his way home, that is $p(1-p)$. So the transition matrix on states $0, \cdots, N$ is given by

$$P = \begin{bmatrix} 1-p & p & 0 & 0 & 0 & \cdots \\ p(1-p) & p^2+(1-p)^2 & (1-p)p & 0 & 0 & \cdots \\ 0 & p(1-p) & p^2+(1-p)^2 & (1-p)p & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & p(1-p) & p^2+(1-p) \end{bmatrix}$$

We need to find the stationary distribution of the chain verifying $\pi P = \pi$. This gives us the following sstem of equations

$$\begin{cases} \pi_0(1-p) + \pi_1 p(1-p) & = \pi_0 \\ \pi_0 + \pi_1(p^2+(1-p)^2) + \pi_2 p(1-p) & = \pi_1 \\ \pi_i(1-p)p + \pi_{i+1}(p^2+(1-p)^2) + \pi_{i+2}p(1-p) & = \pi_{i+1}, 1 \le i \le N-1 \\ \pi_{N-1}p(1-p) + \pi_N(p^2+(1-p)) & = \pi_N. \end{cases}$$

We get from the first equation that $\pi_1 = \frac{\pi_0}{1-p}$. Using this, we get from the second equation that $\pi_2 = \pi_1$. Using this, we get finally that $\pi_1 = \pi_2 = \cdots = \pi_N = \frac{\pi_0}{1-p}$. Since $\sum_{j=0}^{N} \pi_j = 1$, we finally get

$$\pi_0 = \frac{1}{1+\frac{N}{1-p}}, \quad \pi_i = \frac{1}{1-p+N}, \ 1 \le i \le N.$$

If the professor has 0 umbrellas at home, he will get wet if it rains on his way to the office, which has probability p. If he has N umbrellas at home, he will get wet if it doesn't rain on his way to the office and it rains on the way back home (this happens with probability $(1-p)p$). So on the long run, the proportion of journeys on which the professor gets wet is given by

$$\pi_0 p + \pi_N p(1-p) = \frac{2p}{1+\frac{N}{1-p}}.$$

Exercise 5 (**Countable exponential races**). Let I be a countable space and let $T_k, k \in I$, be independent exponential random variables with $T_k \sim Exp(q_k)$ with $0 < q := \sum_{k\in I} q_k < \infty$. Set $T = \inf_k T_k$. Let K be the random variable with values in I that is equal to k whenever $T = T_k$ and $T_j > T_k$ for $j \neq k$. Show that T and K are independent with $T \sim Exp(q)$ and $\mathbb{P}(K = k) = q_k/q$. Deduce that $\mathbb{P}(K = k \text{ for some } k) = 1$.

Solution. We have $K = k$ if $T_k < T_j$ for all $j \neq k$. By the total probability formula, we have

$$\begin{aligned}
\mathbb{P}(K = k \text{ and } T \geq t) &= \mathbb{P}(T_k \geq t \text{ and } T_j > T_k \text{ for all } j \neq k) \\
&= \int_t^\infty q_k e^{-q_k s}\, \mathbb{P}(T_j > s \text{ for all } j \neq k) ds \\
&= \int_t^\infty q_k e^{-q_k s} \prod_{j\neq k} e^{-q_j s} ds \\
&= \int_t^\infty q_k e^{-qs} ds = \frac{q_k}{q} e^{-qt}.
\end{aligned}$$

Hence we have that $\mathbb{P}(K = k \text{ for some } k) = 1$ and T and K have the claimed joint distribution.

Exercise 6 (**General construction of Markov processes**). Let us consider a countable state space E and an array of positive numbers $(\lambda_{i,j})_{i,j\in E; i\neq j}$ with $\sum_{j\in E; j\neq i} \lambda_{i,j} < \infty$ for all $i \in E$. We recursively define a continuous time stochastic process $(X(t))_{t\geq 0}$ on E starting at $i_0 \in E$ as follows:

(i). Define $T_0 = 0$ and set $X(T_0) = i_0 \in E$;

(ii). For $n \in \mathbb{N}$: suppose we know T_{n-1} and $X(T_{n-1}) = i_{n-1}$. Independently of the previous steps, generate independent exponential random variables $E_1, E_2, \dots$ with $E_j \sim Exp(\lambda_{i_{n-1},j})$. Define $T_n = T_{n-1} + \inf_{j\in\mathbb{N}} E_j$ and $i_n = \operatorname{argmin}_{j\in E} E_j$, that is, the (random) index of the exponential variable that is the smallest. Then put

$$X(t) = \begin{cases} i_{n-1} & \text{for } t \in [T_{n-1}, T_n) \\ i_n & \text{for } t = T_n. \end{cases}$$

a) What is the distribution of the time between the jumps of the process $(X(t))_{t\geq 0}$?

b) Let $\widehat{P}_{ij}$ be the probability

$$\widehat{P}_{ij} = \mathbb{P}(X(T_n) = j \mid X(T_{n-1} = i)).$$

Find the matrix $\widehat{P} = (\widehat{P}_{ij})_{i,j\in E}$.

c) Show that $(X(t))_{t\geq 0}$ is a homogeneous Markov process.

Solution. a) We are looking for the distribution of the waiting time between two jumps, i.e. the distribution of $S_n = T_n - T_{n-1}$, by definition this is defined as

$$S_n = \inf_{j\in\mathbb{N}} E_j.$$

According to exercise 1, we have that $S_n \sim Exp(\sum_{j=1}^\infty \lambda_{i_{n-1},j})$.

b) We know by a) that the waiting time between two jumps of the process is a n exponential random variable arising as the minimum of an countable exponential race. The first exercise gives us additionally that

$$\widehat{P}_{ij} = \mathbb{P}(X(T_n) = j \mid X(T_{n-1}) = i) = \frac{\lambda_{i,j}}{\sum_{k \neq i} \lambda_{i,k}}.$$

c) We have to show that $(X(t))_{t \geq 0}$ is a Markov process, that is

$$\mathbb{P}(X_t = j \mid \{X_r, r \leq s, X_s = i\}) = \mathbb{P}(X_t = j \mid X_s = i).$$

Since we condition on $\{X_r, r \leq s, X_s = i\}$, there exists a time m (depending on ω) such that $\{X_r, r \leq s, X_s = i\} = \{(X_r)_{r<s}, T_{m-1} < s < T_m \text{ and } X_s = i\}$. First, note that by construction the process before time T_{m-1} is irrelevant for determining this probability:

$$\begin{aligned}&\mathbb{P}(X_t = j \mid \{X_r, r \leq s \text{ and } T_{m-1} < s < T_m \text{ and } X_s = i\})\\ &= \mathbb{P}(X_t = j \mid \{X_r, T_{m-1} \leq r \leq s \text{ and } T_m > s \text{ and } X_s = i\}).\end{aligned}$$

Then memorylessness property of the exponential random variables implies that for $S_m = T_m - T_{m-1}$

$$S_m \sim S_m - (s - T_{m-1}) \sim Exp(\sum_{j=1}^{\infty} \lambda_{i,j}),$$

i.e. knowing that the exponential rate exceeds $s - T_{m-1}$ is irrelevant for determining the current transitions probabilities. Moreover, since $X_{T_{m-1}} = X_s$ by definition of T_{m-1} and T_m, $\{X_s = i\}$ is the only relevant information for the next evolution of the process based on information contained in $\{X_r, T_{m-1} \leq r \leq s \text{ and } T_m > s \text{ and } X_s = i\}$. Thus

$$\mathbb{P}(X_t = j \mid \{X_r, T_{m-1} \leq r \leq s \text{ and } T_m > s \text{ and } X_s = i\}) = \mathbb{P}(X_t = j \mid X_s = i)$$

this finishes the proof.

Definition (The Q-matrix). One way of thinking about the evolution of the Markov process $(X(t))_{t \geq 0}$ is in terms of its Q-matrix, which is known as the generator of the process. A matrix $Q = (q_{ij})_{i,j \in E}$ is a Q-matrix if it satisfies

(i). $-\infty < q_{ii} \leq 0$ for all $i \in E$;

(ii). $0 \leq q_{ij} < \infty$ for all $i \neq j$;

(iii). $\sum_{j \in E} q_{ij} = 0$ for all $i \in E$.

The Q-matrix of the Markov process $(X(t))_{t \geq 0}$ as constructed above is given by $q_{ii} = -\sum_{j \neq i} \lambda_{i,j}$ for $i \in E$, and $q_{ij} = \lambda_{ij}$ for $j \neq i$.

Exercise 7. In a population of size N, a rumor is begun by a single individual who tells it to everyone he meets; they in turn pass the rumor to everyone they meet, once a person has passed the rumor to somebody he exits the system. Assume that each individual meets another randomly with exponential rate $1/N$. Let $X(t)$, $t \geq 0$ be the number in $E = \{1, \ldots, N\}$ of people who know the rumor at time t.

a) Draw a graph to visualize the chain. Write down the Q-matrix of the chain.

b) How long does it take in average until everyone knows the rumor if $X(0) = 1$?

Solution. a) The Q-matrix has the form

$$\begin{pmatrix} -\frac{N-1}{N} & \frac{N-1}{N} & 0 & \cdots & 0 \\ 0 & -\frac{N-2}{N} & \frac{N-2}{N} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & -\frac{1}{N} & \frac{1}{N} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

b) We need to compute $\mathbb{E}_1(T_N)$ where $T_N = \inf\{t : X(T_N) = N\}$. Remark that T_N is just a sum of exponential random variables

$$T_N = \sum_{i=1}^{N-1} E_i,$$

where $E_i \sim \text{Exp}(\frac{N-i}{N})$, So that

$$E_1(T_N) = \sum_{i=1}^{N-1} \frac{N}{N-i} \approx N \log N.$$

You could notice that this is exactly the continuous time version of the coupon's collector model.

Exercise 8. For $i \in \mathbb{N}$, let E_i be independent copies of an exponential random variable of parameter λ. We let $T_n := E_1 + \cdots + E_n$ and

$$N(t) := \sum_{n=1}^{\infty} \mathbb{1}_{\{T_n \le t\}}, \ t \ge 0.$$

The process $(N(t))_{t\ge 0}$ is called a homogeneous Poisson process with intensity λ. Let $T_0 = 0$ and we say that $T_1, T_2, T_3, \ldots$ are the successive arrival times of the Poisson process, and E_n the intervals $T_n - T_{n-1}$.

(i). Show that T_n follows an Erlang law with parameters n and λ having density:

$$f_{T_n}(t) = \frac{\lambda^n}{(n-1)!} \, t^{n-1} \, e^{-\lambda t} \, \mathbb{1}_{\{t>0\}}.$$

(ii). Show that, $\forall t > 0$, $N(t)$ follows a Poisson law with parameter λt, i.e.

$$\mathbb{P}(N(t) = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \quad k = 0, 1, 2, \ldots.$$

Solution. (i). We proceed by induction on n. For $n = 1$, T_1 follows an exponential law with parameter λ, which is equivalent to an Erlang law of parameter 1 and λ. Suppose that $T_n \sim \text{Erlang}(n, \lambda)$. Remark that $E_{n+1} \sim Exp(\lambda)$ is independent of T_n. For $t > 0$, we have

$$\begin{aligned} F_{T_{n+1}}(t) &= \mathbb{P}(T_{n+1} \leq t) = \mathbb{P}(T_n + E_{n+1} \leq t) = \int_0^\infty \mathbb{P}(T_n + E_{n+1} \leq t \mid T_n = u) f_{T_n}(u) du \\ &= \int_0^t \mathbb{P}(E_{n+1} \leq t - u) f_{T_n}(u) du = \int_0^t F_{E_{n+1}}(t-u) f_{T_n}(u) du. \end{aligned}$$

Implying that

$$\begin{aligned} f_{T_{n+1}}(t) &= \frac{d}{dt} \int_0^t (1 - e^{-\lambda(t-u)}) f_{T_n}(u) du \\ &= \int_0^t \Big((1 - e^{-\lambda(t-u)}) f_{T_n}(u) \Big)' du + (1 - e^{-\lambda(t-t)}) f_{T_n}(t) \\ &= \int_0^t \lambda e^{-\lambda(t-u)} \frac{\lambda^n}{(n-1)!} \; u^{n-1} \; e^{-\lambda u} du \\ &= \frac{\lambda^{n+1}}{(n-1)!} \; e^{-\lambda t} \int_0^t u^{n-1} du \\ &= \frac{\lambda^{n+1}}{n!} \; t^n \; e^{-\lambda t}. \end{aligned}$$

(ii). By definition of $(N(t))_{t \geq 0}$ and of the arrival times T_i, we know that

$$\mathbb{P}(N(t) \geq n) = \mathbb{P}(T_n \leq t) = F_{T_n}(t).$$

By (i), we have

$$\begin{aligned} F_{T_{n+1}}(t) &= \int_0^t (1 - e^{-\lambda(t-u)}) f_{T_n}(u) du = F_{T_n}(t) - \int_0^t e^{-\lambda(t-u)} \cdot \frac{\lambda^n}{(n-1)!} \; u^{n-1} \; e^{-\lambda u} du \\ &= F_{T_n}(t) - \frac{\lambda^n}{n!} \; t^n \; e^{-\lambda t} \end{aligned}$$

It is a recursive relation between $F_{T_{n+1}}(t)$ and $F_{T_n}(t)$. As $F_{T_1}(t) = 1 - e^{-\lambda t}$, we get

$$F_{T_{n+1}}(t) = 1 - \sum_{k=0}^{n} e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \; n \in \mathbb{N}.$$

Using this result, we obtain the distribution of $N(t)$ for a fixed t

$$\begin{aligned} \mathbb{P}(N(t) = n) &= \mathbb{P}(N(t) \geq n) - \mathbb{P}(N(t) \geq n+1) \\ &= \sum_{k=0}^{n} e^{-\lambda t} \frac{(\lambda t)^k}{k!} - \sum_{k=0}^{n-1} e^{-\lambda t} \frac{(\lambda t)^k}{k!} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}. \end{aligned}$$

that is $N(t) \sim Poi(\lambda t)$.