FINAL EXAM

Spring Semester

17 August 2020

Length of the exam : 3h00 (from16h15 to 19h15)

Attempt all the questions

First write your name, given names and section :

Name :______________________________ **Given Name :** ______________________________

Section : _____________________

Exercice	**Points**
1	
2	
3	
4	
Total points :	

1)

a) For a Markov chain on $\mathcal{N} = \{1, 2, 3 \cdots\}$ $q_{i\ i+1} \ = q_i = i^2$. Is the chain explosive ? Justify.

b) An irreducible discrete time Markov chain on $I = \{1, 2, 3, 4\}$ has invariant distribution $(1/4,\ 1/8,\ 3/8,\ 1/4)$. Give the expected number of visits to state 3 by the Markov chain starting at 1, before returning to 1. If we replace the assumption of irreducibility by that of aperiodicity, is the result still true ?

c) Give an example of a continuous time Markov chain $(X_t)_{t\geq 0}$ with an invariant distribution π which is not invariant for the jump chain.

d) For a discrete time Markov chain, if state i leads to state j and i has period 1, must j also have period 1 ? Justify or provide a common example.

e) A continous time Markov chain on $I = \{1, 2, 3\}$ evolves as follows : $\forall i \in I$ while in state i, the chain waits an exponential time of mean $i + 1$ and then selects a 'new' site (which could be i) uniformly among I (so all sites have probability 1/3 of being selected). The chain restarts at the new site and the waiting and jumping begins again. All waiting times and states are conditionally independent. Give Q for the Markov chain.

f) Give an example of a finite state irreducible nonreversible Markov chain.

g) A Poisson process of rate 1, $(X_t)_{t\geq 0}$, with $X_0 = 0$ has $X_2 = 4$. What is the conditional probability that $X_1 = 4$? How does this change if the rate is changed to 3.

h) For two independent exponential $r.v.s.$ X, Y of parameter λ and μ respectively, let $U = \min\{X, Y\}$ $V = \max\{X, Y\}$. What is the law of U ? Are U and $V - U$ independant ?

i) Let λ be a probability on $I = \{1, 2 \cdots\ N\}$. Let transition probability P satisfy $P_{ij} = \lambda_j\ \forall i, j$. Give a necessary and sufficient condition for the chain to be irreducible. With this condition is it aperiodic ? Reversible with respect to λ ?

.

.

2)

Consider a discrete time Markov chain with transition matrix on $I = \{1, 2, 3, 4, 5\}$

$$P = \begin{pmatrix} 1/4 & 0 & 1/2 & 1/4 & 0 \\ 1/4 & 0 & 1/2 & 0 & 1/4 \\ 1/4 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

a) Find the probability of reaching 4 starting from 2 for the Markov chain.

b) Let $h(x) = E(\sum_{n=0}^{\infty} I_{X_n=1} \mid X_0 = x)$
Calculate $h(2)$

c) For an irreducible Markov chain on (finite or countable) state space I, let i and j be distinct sites and define
$h(x) = E\left(\sum_{n=0}^{T_j} I_{X_n=i} \mid X_0 = x\right)$

where $T_j = \inf\{n \geq 0 : \quad X_n = j\}$

(i) Show $h(x) < \infty \quad \forall x \in I$. Is this so if the chain is no longer ireducible ?

(ii) Give a characterization (via a system of equations) for h without proof.

.

3)

Consider a Markov chain X on $\mathcal{N} = \{1, 2, 3 \cdots\}$
with $P_{i\ i+1} = \left(\frac{i}{i+1}\right)^2 \quad P_{i\ 1} = 1 - \left(\frac{i}{i+1}\right)^2$

a) Show the chain is irreducible. Is it periodic ?

b) Show the chain is positive recurrant and give the stationary distribution π. Is the chain reversible for π ?

c) What can you say about the number of times that the chain has visited 5 by time 10^6 if $X_1 = 1$.

d) If we change $P_{i\ i+1}$ to $\left(\frac{i}{i+1}\right)^\alpha$ and $P_{i\ 1} = 1 - \left(\frac{i}{i+1}\right)^\alpha$,where $\alpha > 0$, for which α is the chain recurrent, for which does it have an invariant distribution ?

.

4)

Let $(N_t)_{t\geq 0}$ be a rate λ Poisson process starting at 0.

Let $\{K_i\}_{i=1}$ be *i.i.d.* Bernoulli(p) *r.v.s* (independant of N). If the jump times of N are at $\{0 < t_1 < t_2 \ \cdots\} = T$,
let $V = \{t_i : \ K_i = 1\}$
and let $(N^1)_{t\geq 0}$ be the process on $\mathcal{N} \cup \{0\}$ that start at 0 and jumps (by 1) at times in V

Define $(N_t^2)_{\geq 0}$ by $N_t^2 = N_t - N_t^1$

a) What are distributions of N^1 and N^2 Are they independant ?

b) Using part a) show that if $e_1, e_2, \cdots$ are *i.i.d.* $\mathcal{E}xp(\lambda)$ and $K_1, K_2 \cdots$ are *i.i.d.* Bernoulli (p) then $\sum_{1=1}^{M_1} e_i$ is $\mathcal{E}xp(\lambda p)$ for $M_1 = \inf\{n \geq 1 \ K_n = 1\}$. If $M_2 = \inf\{n \geq 1 \ K_n = 0\}$ are $\sum_{1=1}^{M_1} e_i$ and $\sum_{1=1}^{M_2} e_i$ independent ?

c) For $N(t)$ consider $S_i = t_{3i}$ for $i = 1, 2 \cdots$ where again t_i are jump times of N. What is the is asymptotic distribution of $S_{j(t)+1} - S_{j(t)}$ as $t \to \infty$ where $j(t) = \max\{j : \ S_j \leq t\}$

.

.