Solution to Problem Set 9 April 15, 2025

Exercise 1.

We begin by noting that the event $\{B_t = x\}$ has probability zero for $t > 0$. Thus, we will use in events weak or strict inequalities interchangeably.

(a) The event $\{T_t \leqslant t_1\}$ is the set of paths that visit 0 between t and t_1. Calculating the probability of this event amounts to calculating the probability of visiting 0 at least once between t and t_1. In other words, thanks to a relation from the lecture, we have

$$\begin{aligned}\mathbb{P}\{T_t \leqslant t_1\} &= \mathbb{P}\{\exists s \in (t, t_1) : B_s = 0\}\\ &= \mathbb{P}\{\exists s \in (t, t_1) : B_s = 0 \mid B_0 = 0\}\\ &= \frac{2}{\pi} \arccos \sqrt{\frac{t}{t_1}}.\end{aligned}$$

(b) The event $\{L_t < t_0,\ T_t > t_1\}$ is the set of paths such that the last visit to 0 before t occurs before t_0 and the next visit to 0 occurs after t_1. This amounts to calculating the probability of the event

$$\{\text{no visit to 0 between } t_0 \text{ and } t_1\} \cap \{\text{at least one visit to 0 before } t_0\}.$$

Since $B_0 = 0$, the event $\{\text{at least one visit to 0 before } t_0\} = \{L_{t_0} < t_0\}$ is a sure event. Consequently, using a result from the lecture and the fact that $\arccos(x) + \arcsin(x) = \frac{\pi}{2}$ for every real number x, we find that

$$\begin{aligned}\mathbb{P}\{L_t < t_0,\ T_t > t_1\} &= \mathbb{P}\{\text{no visit to 0 between } t_0 \text{ and } t_1\}\\ &= 1 - \mathbb{P}\{\exists t \in (t_0, t_1) : B_t = 0\}\\ &= 1 - \frac{2}{\pi} \arccos \sqrt{\frac{t_0}{t_1}}\\ &= \frac{2}{\pi} \arcsin \sqrt{\frac{t_0}{t_1}}.\end{aligned}$$

Exercise 2.

The calculation is straightforward. We have

$$\begin{aligned}\mathbb{E}[U_t U_s] &= e^{-t-s}\mathbb{E}[B_{e^{2t}} B_{e^{2s}}]\\ &= e^{-t-s}(e^{2t} \wedge e^{2s})\\ &= e^{2(t\wedge s)-t-s}\\ &= e^{-|s-t|}.\end{aligned}$$

The analogous calculation for V_t is done directly as well. We have

$$\begin{aligned}\mathbb{E}[V_tV_s] &= \mathbb{E}[(B_t - tB_1)(B_s - sB_1)] \\ &= \mathbb{E}[B_tB_s] - t\,\mathbb{E}[B_1B_s] - s\,\mathbb{E}[B_tB_1] + st\,\mathbb{E}[B_1^2] \\ &= (t\wedge s) - ts - ts + st \\ &= (t\wedge s)(1 - t\vee s).\end{aligned}$$

Notice that if $s = t$, then $\mathbb{E}[V_t^2] = t(1-t)$. Moreover, the process V_t is Gaussian and centered, that is, we recover the conditional probability from Exercise 3 of Series 8 with $a = b = 0$ and $t_1 = 0,\ t_2 = 1$.

Exercise 3.

By Fubini's theorem, we have

$$\mathbb{E}\left[\int_0^t B_s\,ds\right] = \int_0^t \mathbb{E}[B_s]\,ds = 0.$$

For the calculation of the variance, we also use Fubini's theorem to find

$$\begin{aligned}\mathbb{E}\left[\int_0^t B_s\,ds \int_0^t B_u\,du\right] &= \int_0^t\int_0^t \mathbb{E}[B_sB_u]\,ds\,du \\ &= \int_0^t\int_0^t (u\wedge s)\,ds\,du \\ &= \int_0^t \left(\int_0^u (u\wedge s)\,ds + \int_u^t (u\wedge s)\,ds\right) du \\ &= \int_0^t \left[\frac{u^2}{2} + u(t-u)\right] du \\ &= \left[-\frac{u^3}{6}\right]_0^t + t\left[\frac{u^2}{2}\right]_0^t \\ &= -\frac{t^3}{6} + \frac{t^3}{2} \\ &= \frac{t^3}{3}.\end{aligned}$$

Exercise 4.

Use that $g_n = g_{Z_n} = g^{\circ n}$, by Taylor formula we have

$$\begin{aligned}\mathbb{P}[Z_{n+1} > 0] &= 1 - g_{n+1}(0) = 1 - g(1 - (1 - g_n(0))) \\ &\overset{\text{Taylor}}{=} 1 - \left(g(1) + g'(1)\big(-1 + g_n(0)\big) + \frac{g''(\xi)}{2}(1 - g_n(0))^2\right) \\ &= m(1 - g_n(0)) - \underbrace{\frac{g''(\xi)}{2}}_{>0}(1 - g_n(0))^2\end{aligned}$$

for some $\xi \in [g_n(0), 1]$. Note that $g''(\xi)$ is bounded in $[0,1]$ as the second moment of the reproduction law μ is bounded. Hence there exists a constant $C \in (0,\infty)$ (uniform in n), such that

$$m(1-g_n(0)) - C(1-g_n(0))^2 \leqslant 1 - g_{n+1}(0) \leqslant m(1-g_n(0)). \tag{1}$$

Moreover, we obtain for all $n \geqslant 0$, $1-g_n(0) \leqslant \ldots \leqslant m^{n-1}(1-g_1(0)) \leqslant m^n$. By rewriting (1) we have

$$1 - \frac{C}{m}(1-g_n(0)) \leqslant \frac{(1-g_{n+1}(0))m^{-1}}{(1-g_n(0))m} = \frac{(1-g_{n+1}(0))m^{-n-1}}{(1-g_n(0))m^{-n}} \leqslant 1;$$

and furthermore, note that $1 - Cm^{n-1} \leqslant 1 - \frac{C}{m}(1-g_n(0))$, thus we have

$$1 - Cm^{n-1} \leqslant \underbrace{\frac{m^{-n-1}\mathbb{P}[Z_{n+1} > 0]}{m^{-n}\mathbb{P}[Z_n > 0]}}_{r_n} \leqslant 1. \tag{2}$$

Note that as n tends to infinity, both sides of the inequality converge to 1. Moreover, since for $n = 0$, $m^{-n}\mathbb{P}[Z_n > 0] = 1$, $m^{-N-1}\mathbb{P}[Z_{N+1} > 0] = \prod_{n=0}^{N} r_n$. By a well-known criterion for convergence of infinite products, $m^{-N-1}\mathbb{P}[Z_{N+1} > 0]$ converges if and only if $\sum_n \log(r_n)$ converges, but the latter is true since $m < 1$ and thus $\sum_n Cm^{-n-1}$ converges (we have also used that from above $\sum_n \log(r_n)$ is bounded by 0). So we have, as desired, $\lim_n m^{-n}\mathbb{P}[Z_n > 0] = c_* \in (0,1]$, and $\mathbb{P}[Z_n > 0] \sim c_* m^n$ as $n \to \infty$.