Solution to Problem Set 8 April 8, 2025

Exercise 1.

(a) T is the first time at which the process dies out and Y is the total number of individuals over time. There is a finite number of individuals if and only if the process dies out at some time, hence the equality of the probabilities.

(b) We must first show that $g_Y(s) = s\, g_Z(g_Y(s))$. Since $X_0 = 1$ almost surely,

$$\begin{aligned} g_Y(s) &= \mathbb{E}[s^Y 1_{\{Y<\infty\}}] = \mathbb{E}[s^{X_0+X_1+\cdots} \cdot 1_{\{Y<\infty\}}] = s\,\mathbb{E}[s^{X_1+\cdots} \cdot 1_{\{Y<\infty\}}] \\ &= s \sum_{k=0}^{+\infty} \mathbb{E}[s^{X_1+\cdots} \cdot 1_{\{Y<\infty\}} \mid X_1 = k]\, \mathbb{P}(X_1 = k) \\ &= s \sum_{k=0}^{+\infty} \mathbb{E}[s^{Y_1+\cdots+Y_k} \cdot \prod_{i=1}^{k} 1_{\{Y_i<\infty\}}]\, \mathbb{P}(Z = k) \end{aligned}$$

where Y_i is the number of descendants of the ith child of X_0, for each child i of X_0. These variables $Y_1, \ldots, Y_k$ are independent and identically distributed, thus we have

$$\begin{aligned} g_Y(s) &= s \sum_{k=0}^{+\infty} (g_Y(s))^k\, \mathbb{P}(Z = k) \\ &= s\, g_Z(g_Y(s)). \end{aligned}$$

Finally, we must verify that the solution is unique. There are three cases to discuss:

1. $s = 0$. Then $x = 0$, and moreover $g_Y(0) = \mathbb{P}(Y = 0) = 0$ since $Y \geqslant X_0 = 1$.

2. $s = 1$. Then we have $x = g_Z(x)$, and by the extinction theorem, the smallest solution of this equation is the extinction probability α. But since we are looking for solutions of the equation in the interval $[0, \alpha]$, the solution is indeed unique. Also note that

$$g_Y(1) = \sum_{k=0}^{+\infty} \mathbb{P}(Y = k) = \mathbb{P}(Y < +\infty) = \mathbb{P}(T < +\infty) = \alpha.$$

3. $s \in]0, 1[$. One must study the function $x \in [0, 1] \mapsto s\, g_Z(x)$. The function is continuous, strictly convex, and increasing. Moreover, at 0, it takes the value $s\, p_0 \in [0, 1]$, and at 1, it equals $s \in]0, 1[$. All this implies that the solution of

$$x = s\, g_Z(x)$$

is unique.

Furthermore, since $s \in [0,1]$, we notice that the function $x \mapsto s\, g_Z(x)$ is less than or equal to the function $x \mapsto g_Z(x)$ for every x. Now, since the solutions of the equation $x = g_Z(x)$ are α and 1, we can deduce that the solutions of the equation $x = s\, g_Z(x)$ are either less than or equal to α or greater than or equal to 1. Thus, if this equation admits a solution in the interval $[0,1]$, it must necessarily lie in the interval $[0,\alpha]$.

Exercise 2.

According to the lecture, we have the joint densities

$$f_{B_{t_1},\ldots,\,B_{t_n}}(x_1,\ldots,\,x_n) = p(x_1,t_1)p(x_2-x_1,t_2-t_1)\cdots p(x_n-x_{n-1},t_n-t_{n-1})$$

and

$$f_{B_{t_1},\ldots,B_{t_n},\,B_t}(x_1,\ldots,\,x_n,y) = p(x_1,t_1)p(x_2-x_1,t_2-t_1)\cdots p(x_n-x_{n-1},t_n-t_{n-1})p(y-x_n,t-t_n).$$

We then obtain the conditional density

$$\begin{aligned} f_{B_t|B_{t_1},\ldots,\,B_{t_n}}(y \mid x_1,\ldots,\,x_n) &= \frac{f_{B_{t_1},\ldots,\,B_{t_n},\,B_t}(x_1,\ldots,\,x_n,y)}{f_{B_{t_1},\ldots,\,B_{t_n}}(x_1,\ldots,\,x_n)} \\ &= p(y-x_n,t-t_n)\cdot\frac{p(x_n,t_n)}{p(x_n,t_n)} \\ &= \frac{f_{B_{t_n},\,B_t}(x_n,y)}{f_{B_{t_n}}(x_n)} = f_{B_t|B_{t_n}}(y \mid x_n). \end{aligned}$$

We therefore have the Markov property:

$$\mathbb{P}\{B_t \leqslant y \mid B_{t_n} = x_n,\ldots,\,B_{t_1} = x_1\} = \mathbb{P}\{B_t \leqslant y \mid B_{t_n} = x_n\}.$$

Exercise 3.

Using the same notations as in the previous exercise, we have

$$\begin{aligned} f_{B_t|B_{t_1},B_{t_2}}(y \mid a,b) &= \frac{f_{B_{t_1},\,B_t,\,B_{t_2}}(a,y,b)}{f_{B_{t_1},B_{t_2}}(a,b)} \\ &= \frac{p(a,t_1)p(y-a,t-t_1)p(b-y,t_2-t)}{p(a,t_1)p(b-a,t_2-t_1)} \\ &= \frac{p(y-a,t-t_1)p(b-y,t_2-t)}{p(b-a,t_2-t_1)} \end{aligned}$$

where $p(x,t) = \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{x^2}{2t}}$. We then obtain

$$\begin{aligned}
f_{B_t|B_{t_1},B_{t_2}}(y \mid a,b) &= \frac{\sqrt{2\pi(t_2-t_1)}\,\exp\left(-\frac{(y-a)^2}{2(t-t_1)}\right)\exp\left(-\frac{(b-y)^2}{2(t_2-t)}\right)}{\sqrt{2\pi(t-t_1)}\,\sqrt{2\pi(t_2-t)}\,\exp\left(-\frac{(b-a)^2}{2(t_2-t_1)}\right)} \\
&= \frac{1}{\sqrt{2\pi\frac{(t-t_1)(t_2-t)}{(t_2-t_1)}}}\,\cdot \\
&\quad \exp\left(-\frac{y^2-2y\left(a\frac{t_2-t}{t_2-t_1}+b\frac{t-t_1}{t_2-t_1}\right)+\left(a^2\frac{t_2-t}{t_2-t_1}+b^2\frac{t-t_1}{t_2-t_1}-(b-a)^2\frac{(t-t_1)(t_2-t)}{(t_2-t_1)^2}\right)}{2\frac{(t-t_1)(t_2-t)}{t_2-t_1}}\right) \\
&= \frac{1}{\sqrt{2\pi\frac{(t-t_1)(t_2-t)}{(t_2-t_1)}}}\exp\left(-\frac{\left\{y-\left(a+(b-a)\frac{t-t_1}{t_2-t_1}\right)\right\}^2}{2\frac{(t-t_1)(t_2-t)}{t_2-t_1}}\right)
\end{aligned}$$

which corresponds to the density of the law

$$\mathcal{N}\left(a+(b-a)\frac{t-t_1}{t_2-t_1},\ \frac{(t-t_1)(t_2-t)}{t_2-t_1}\right).$$

Exercise 4.

The calculation is straightforward using the independence of increments and the fact that B_t follows a $\mathcal{N}(0,t)$ distribution. If $s \leqslant t$, we obtain

$$\begin{aligned}
\mathbb{E}[B_sB_t] &= \mathbb{E}\Big[B_s\Big((B_t-B_s)+B_s\Big)\Big] \\
&= \mathbb{E}[B_s(B_t-B_s)]+\mathbb{E}[B_s^2] \\
&= \mathbb{E}[B_s]\mathbb{E}[B_t-B_s]+s \\
&= 0+s \\
&= s=\min(s,t).
\end{aligned}$$

Exercise 5.

(a) To show this we simply need to check that for any $n \geqslant 1$, $n \in \mathbb{N}^*$, we have

$$\mathbb{E}[Z_n|Y_{n+1}] = Z_{n+1}.$$

In fact,

$$\mathbb{E}[Z_n|Y_{n+1}] = \mathbb{E}[\mathbb{E}[X|Y_n]|Y_{n+1}] = \mathbb{E}[\mathbb{E}[X|X_n,Y_{n+1}]|Y_{n+1}] = \mathbb{E}[X|Y_{n+1}] = Z_{n+1}.$$

(b) For any $a<b$ and any $N \geqslant 1$, the upcrossing lemma for martingales in (a) implies that

$$\mathbb{E}[U_{a,b,N}] \leqslant \frac{1}{b-a}(\mathbb{E}[(Z_1-a)_+]-\mathbb{E}[(Z_N-a)_+]) \leqslant \frac{1}{b-a}\mathbb{E}[(Z_1-a)_+].$$

where $U_{a,b,N}$ is the number of upcrossings of the interval $[a,b]$ between 1 and N for the martingale $(Z_{N-k+1}, k=1,\dots,N)$. Note that

$$\mathbb{E}[(Z_1-a)_+] \leqslant a + \mathbb{E}[|Z_1|] = a + \mathbb{E}[|\mathbb{E}[X \mid Y_1]|] \leqslant a + \mathbb{E}[\mathbb{E}[|X| \mid Y_1]] = a + \mathbb{E}[|X|].$$

If we define $U_{a,b} := \lim_{N\to\infty} U_{a,b,N}$ to be the number of upcrossings(downcrossings) of $[a,b]$ by $(Z_k, k\in\mathbb{N}^*)$, by Monotone convergence theorem we have $\mathbb{E}[U_{a,b}] = \lim_{N\to\infty}\mathbb{E}[U_{a,b,N}] \leqslant \frac{1}{b-a}(a+\mathbb{E}[|X|])$, which is finite. Thus $U_{a,b}$ is a.s. finite.

(c) As the a.s. convergence theorem of submartingale, let $a = \liminf_{k\to\infty} Z_k$, $b = \limsup_{k\to\infty} Z_k$, if $a \neq b$ we can check that $U_{a,b}$ is not finite, which gives a contradiction.