Solution to Problem Set 7 April 1, 2025

Exercise 1.

(a) For every $s \in [-1,1]$, we have

$$\left|\sum_{j=0}^{+\infty} p_j s^j\right| \leqslant \sum_{j=0}^{+\infty} |p_j s^j| = \sum_{j=0}^{+\infty} p_j |s^j| \leqslant \sum_{j=0}^{+\infty} p_j = 1,$$

and thus the series in the statement is dominated by a convergent series that does not depend on s. Therefore, the series converges uniformly.

(b) Since Z is a discrete random variable, for every $s \in [-1,1]$ we have

$$\mathbb{E}[s^Z] = \sum_{j=0}^{+\infty} s^j \, \mathbb{P}\{Z = j\} = \sum_{j=0}^{+\infty} p_j s^j = g_Z(s).$$

(c) Using (b) above and the independence of Z and Z', it follows that

$$g_{Z+Z'}(s) = \mathbb{E}[s^{Z+Z'}] = \mathbb{E}[s^Z s^{Z'}] = \mathbb{E}[s^Z]\mathbb{E}[s^{Z'}] = g_Z(s) g_{Z'}(s).$$

(d) Since $\sum_{j=0}^{+\infty} j p_j < +\infty$, by dominated convergence theorem we can differentiate term by term, which gives

$$g_Z'(s) = \sum_{j=1}^{+\infty} j p_j s^{j-1}$$

and

$$g_Z'(1) = \sum_{j=1}^{+\infty} j p_j = \sum_{j=1}^{+\infty} j \, \mathbb{P}(Z = j) = \mathbb{E}[Z].$$

Similarly, note that if $\sum_{j=0}^{+\infty} j^2 p_j < +\infty$, then $\sum_{j=0}^{+\infty} j p_j$ converges, as does $\sum_{j=0}^{+\infty} j(j-1)p_j$. Consequently,

$$g_Z''(s) \quad = \quad \sum_{j=2}^{+\infty} j(j-1) p_j s^{j-2}$$

and

$$g_Z''(1) = \sum_{j=2}^{+\infty} j(j-1) p_j = \mathbb{E}[Z(Z-1)].$$

Since $\mathrm{Var}(Z) = \mathbb{E}[Z^2] - \mathbb{E}[Z]^2$, we then obtain

$$\begin{aligned} \mathrm{Var}(Z) &= \mathbb{E}[Z(Z-1)] + \mathbb{E}[Z] - \mathbb{E}[Z]^2 \\ &= g_Z''(1) + \mathbb{E}[Z] - \mathbb{E}[Z]^2. \end{aligned}$$

Exercise 2.

Given that $\alpha \leqslant 1$, we have

$$\mathbb{E}[|S_n|] = \mathbb{E}[\alpha^{X_n}] = \sum_{j \geqslant 0} \mathbb{P}(X_n = j)\alpha^j \leqslant 1.$$

Recall that α is the solution of the equation $g_Z(s) = s$. Suppose $X_0 = i_0, \ldots, X_n = i_n$ and denote by $Z_1, \ldots, Z_{i_n}$ the i_n random variables whose sum defines X_{n+1}. Then, using the fact that (X_n) is a branching process, that the random variables $Z_1, \ldots, Z_{i_n}$ are mutually independent and also independent of X_n, and that they share the same distribution, we obtain

$$\begin{aligned}
\mathbb{E}[S_{n+1} \mid X_0 = i_0, \ldots, X_n = i_n] &= \mathbb{E}[\alpha^{X_{n+1}} \mid X_0 = i_0, \ldots, X_n = i_n] \\
&= \mathbb{E}[\alpha^{Z_1 + \cdots + Z_{i_n}} \mid X_0 = i_0, \ldots, X_n = i_n] \\
&= \mathbb{E}[\alpha^{Z_1 + \cdots + Z_{i_n}}] \\
&= \prod_{i=1}^{i_n} \mathbb{E}[\alpha^{Z_i}] \\
&= \big(g_Z(\alpha)\big)^{i_n} = \alpha^{i_n},
\end{aligned}$$

which means that $\mathbb{E}[S_{n+1} \mid X_0, \ldots, X_n] = \alpha^{X_n} = S_n$. This proves the result.

Exercise 3.

This problem can be modeled using a branching process. Suppose that at time 1, there are Z_1 customers in line. They form generation 1. The direct descendants of a customer are those who arrive while that customer is being served. Generation $n+1$ is formed by the direct descendants of the customers in generation n. Thus, each of the X_n customers in generation n is served during one minute, during which a certain number Z_i (for $i = 1, \ldots, X_n$) of customers arrive. These are the direct descendants. Once the X_n customers of generation n have been served, we are left with $X_{n+1} = Z_{t_n+1} + \cdots + Z_{t_n+X_n}$ new customers in line. Here, t_n is the time spent in the generations before. Hence, X_n is a branching process.

We thus see that the probability that the server can take a break is equal to the extinction probability α of the process (X_n). Indeed, if $X_n = 0$, then there are no more customers in line.

There are three cases to consider.

- If there exists some $j \geqslant 1$ such that $p_j = 1$, then the extinction probability is zero.
- If $p_1 < 1$ and $p_0 + p_1 = 1$, then, by the continuity of probabilities,

  $$\mathbb{P}\{X_n \geqslant 1 \; \forall\, n \geqslant 1\} = \lim_{m \to \infty} \mathbb{P}\{X_n \geqslant 1 \; \forall\, 1 \leqslant n \leqslant m\} \leqslant \limsup_{m \to \infty} \big(1 - p_0^{X_0}\big)^m = 0,$$

  and therefore the extinction probability is 1.

- Finally, if $p_j < 1$ for all $j \geqslant 0$ and $p_0 + p_1 < 1$, then the assumptions A and B of the course are satisfied, and hence the extinction probability is the smallest solution of the equation $g_Z(s) = s$ (one can verify that this holds even in the first two cases).

Numerical application: We seek the solutions of $\alpha = \frac{1}{5} + \frac{1}{5}\alpha + \frac{3}{5}\alpha^2$. We find $\alpha = 1$ or $\alpha = \frac{1}{3}$, with the latter being the smallest solution. Hence, the probability that the server can take a break is $\frac{1}{3}$.

Exercise 4.

(a) Since $-X_n$ is a submartingale, the a.s.-submartingale convergence theorem tells us that $-X_n$ converges, which means that X_n converges a.s. as well.

(b) 2 is just the a.s.-submartingale convergence theorem. For 1, non-negative means that $\mathbb{E}[X_n^-] = 0 < \infty$, thus we have X_n converges a.s. by part (a). 3 is similar to 1.

Exercise 5.

1. Let $(S_n)_n$ be a SRW on $\mathbb{Z}$ and τ_1 its first hitting time of 1. We saw that $(S_{n\wedge\tau_1})_n$ is a martingale, which almost surely converges to $S_{\tau_1} = 1$. Latter follows from the fact that τ_1 is almost surely finite, which we have seen in class. However, for all n,

$$0 = \mathbb{E}[S_0] = \lim_n \mathbb{E}[S_{n\wedge\tau_1}] \neq \mathbb{E}[S_{\tau_1}] = 1.$$

   Therefore, $(S_{n\wedge\tau_1})_n$ can not converge in L^1.

2. Let $(X_n)_n$ be iid standard normal random variables. Clearly, $\sup_n \mathbb{E}[|X_n|] = \mathbb{E}[|X_n|] \leqslant 1$. And by Borel-Cantelli II applied to $(E_n(C) = \{X_n > C\})_n$,

$$\mathbb{P}[\limsup_n X_n \geqslant C] \geqslant \mathbb{P}[\limsup_n E_n] = 1.$$

   This holds for any $C > 0$. Therefore, when taking $C \uparrow \infty$ the monotonicity of the probability measure implies that $\limsup_n X_n = \infty$ almost surely.[1]

[1] A more general statement is the following: let $(X_n)_n$ be iid random variables with c.d.f. F, then $\limsup_n X_n = M$ almost surely, where $M = \inf\{t \in \mathbb{R} : F(t) = 1\}$.