Solution to Problem Set 6 March 25, 2025

Exercise 1.

First, let us show that X_n converges to 0 in probability. Let $\varepsilon > 0$. Then

$$\mathbb{P}\{|X_n - 0| > \varepsilon\} = \mathbb{P}\{X_n = 1\} = \frac{1}{n} \underset{n\to+\infty}{\longrightarrow} 0.$$

Now, let us show that X_n does not converge almost surely. First, note that if the almost sure limit exists, it must be 0. Indeed, by contradiction, if X_n converges almost surely to a random variable $X \neq 0$, then X_n also converges to X in probability, which contradicts the previous result.

Recall the following result from the lecture:

$$X_n \overset{\text{a.s.}}{\to} 0 \qquad \Leftrightarrow \qquad \forall\varepsilon > 0, \lim_{n\to+\infty} \mathbb{P}\left(\bigcup_{m\geqslant n} \{|X_m - 0| > \varepsilon\}\right) = 0. \tag{1}$$

Thus, it is enough to show that there exists $\varepsilon > 0$ such that the right-hand side of (1) does not hold. In fact, we will show that for all $\varepsilon > 0$, the above limit equals 1.

Let $\varepsilon \in (0,1)$. By definition of X_m and the pairwise independence of X_m, for all $n \geqslant 1$,

$$\begin{aligned}\mathbb{P}\left(\bigcap_{m\geqslant n} \{|X_m - 0| \leqslant \varepsilon\}\right) &= \prod_{m\geqslant n} \mathbb{P}\{X_m = 0\} \\ &= \prod_{m\geqslant n} \left(1 - \frac{1}{m}\right) \\ &= \lim_{N\to+\infty} \prod_{m=n}^{N} \frac{m-1}{m} \\ &= \lim_{N\to+\infty} \frac{n-1}{N} \;=\; 0.\end{aligned}$$

This implies that for all $\varepsilon > 0$,

$$\lim_{n\to+\infty} \mathbb{P}\left(\bigcap_{m\geqslant n} \{|X_m| \leqslant \varepsilon\}\right) = 0,$$

or equivalently, by taking the complement,

$$\lim_{n\to+\infty} \mathbb{P}\left(\bigcup_{m\geqslant n} \{|X_m - 0| > \varepsilon\}\right) = 1.$$

Thus, X_n does not converge almost surely to 0, so it does not converge almost surely.

Exercise 2.

We will begin by proving the following result, which is an extension of Doob-Kolmogorov's inequality: if (S_n) is a submartingale (relative to (X_n)) and $\lambda > 0$, then

$$\mathbb{P}\left(\max_{1\leqslant k\leqslant n} S_k \geqslant \lambda\right) \leqslant \frac{1}{\lambda}\,\mathbb{E}\left[S_n \mathbb{1}_{\{\max\limits_{1\leqslant k\leqslant n} S_k \geqslant \lambda\}}\right]. \tag{2}$$

Indeed, define

$$T = \begin{cases} \inf\{i \geqslant 0 : S_i \geqslant \lambda\} & \text{if } \max\limits_{1\leqslant k\leqslant n} S_k \geqslant \lambda \\ n & \text{if } \max\limits_{1\leqslant k\leqslant n} S_k < \lambda. \end{cases}$$

Then T is a stopping time relative to (X_n), finite almost surely. Let the event $A = \{\max\limits_{1\leqslant k\leqslant n} S_k \geqslant \lambda\}$. Then, $T \leqslant n$ and on A, $S_T \geqslant \lambda$. On A^c, $S_n = S_T$. Applying the stopping theorem, we obtain

$$\begin{aligned} \mathbb{E}[S_n \mathbb{1}_A] &= \mathbb{E}[S_n(1 - \mathbb{1}_{A^c})] = \mathbb{E}[S_n] - \mathbb{E}[S_n \mathbb{1}_{A^c}] = \mathbb{E}[S_n] - \mathbb{E}[S_T \mathbb{1}_{A^c}] \\ &\geqslant \mathbb{E}[S_T] - \mathbb{E}[S_T \mathbb{1}_{A^c}] = \mathbb{E}[S_T \mathbb{1}_A] \geqslant \mathbb{E}[\lambda \mathbb{1}_A] = \lambda \mathbb{P}(A). \end{aligned}$$

Thus, the result is proved. Consequently, the inequality

$$\mathbb{P}\left(\max_{1\leqslant k\leqslant n} |S_k| \geqslant \lambda\right) \leqslant \frac{1}{\lambda}\,\mathbb{E}\left[|S_n| \mathbb{1}_{\{\max\limits_{1\leqslant k\leqslant n} |S_k| \geqslant \lambda\}}\right], \tag{3}$$

valid for any martingale (S_n) and any $\lambda > 0$, is a consequence of (2), applied to the submartingale $(|S_n|)$.

Now, we will analyze the quantity $\mathbb{E}[(\max_{1\leqslant k\leqslant n} |S_k| \wedge M)^p]$, with $M \in \mathbb{R}$ fixed. This ensures that the expectation is finite. Moreover, we observe that the event $\{\max_{1\leqslant k\leqslant n} |S_k| \wedge M \geqslant y\}$ is equal to $\{\max_{1\leqslant k\leqslant n} |S_k| \geqslant y\}$ if $y \leqslant M$ and to $\emptyset$ otherwise. Thus, we can apply inequality (3) when $y \leqslant M$. Using this inequality, Fubini's theorem, and Hölder's inequality (with $\frac{1}{p} + \frac{1}{\frac{p}{p-1}} = 1$), we obtain

$$\begin{aligned} \mathbb{E}\left[\left(\max_{1\leqslant k\leqslant n} |S_k| \wedge M\right)^p\right] &= \int_0^{+\infty} p y^{p-1} \mathbb{P}\left(\max_{1\leqslant k\leqslant n} |S_k| \wedge M \geqslant y\right) dy \\ &= \int_0^{M} p y^{p-1} \mathbb{P}\left(\max_{1\leqslant k\leqslant n} |S_k| \geqslant y\right) dy \\ &\leqslant \int_0^{M} p y^{p-2} \mathbb{E}\left[|S_n| \mathbb{1}_{\left\{\max\limits_{1\leqslant k\leqslant n} |S_k| \geqslant y\right\}}\right] dy \\ &= \int_0^{+\infty} p y^{p-2} \mathbb{E}\left[|S_n| \mathbb{1}_{\left\{\max\limits_{1\leqslant k\leqslant n} |S_k| \wedge M \geqslant y\right\}}\right] dy \\ &= \mathbb{E}\left[\int_0^{+\infty} p y^{p-2} |S_n| \mathbb{1}_{\left\{\max\limits_{1\leqslant k\leqslant n} |S_k| \wedge M \geqslant y\right\}}\, dy\right] \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left[|S_n|\left(\int_0^{\max\limits_{1\leqslant k\leqslant n}|S_k|\wedge M} p y^{p-2}\,dy\right)\right] \\
&= \mathbb{E}\left[|S_n|\frac{p}{p-1}\left(\max_{1\leqslant k\leqslant n}|S_k|\wedge M\right)^{p-1}\right] \\
&= \frac{p}{p-1}\,\mathbb{E}\left[|S_n|\left(\max_{1\leqslant k\leqslant n}|S_k|\wedge M\right)^{p-1}\right] \\
&\leqslant \frac{p}{p-1}\,(\mathbb{E}[|S_n|^p])^{\frac{1}{p}}\left(\mathbb{E}\left[\left(\max_{1\leqslant k\leqslant n}|S_k|\wedge M\right)^{p}\right]\right)^{\frac{p-1}{p}}
\end{aligned}
$$

Moving the last factor on the right-nand side to the left, we find that

$$
\mathbb{E}\left[\left(\max_{1\leqslant k\leqslant n}|S_k|\wedge M\right)^{p}\right]^{\frac{1}{p}} \leqslant \frac{p}{p-1}(\mathbb{E}[|S_n|^p])^{\frac{1}{p}}.
$$

Letting $M \to \infty$ and applying the monotone convergence theorem, we obtain

$$
\mathbb{E}\left[\left(\max_{1\leqslant k\leqslant n}|S_k|\right)^{p}\right]^{\frac{1}{p}} \leqslant \frac{p}{p-1}(\mathbb{E}[|S_n|^p])^{\frac{1}{p}}.
$$

Exercise 3.

Set $M_n := \sum_{i=1}^n a_i X_i$. Check that $(M_n)_n$ is a martingale. Note that

$$
\mathbb{E}[M_n^2] = \sum_{i=1}^n a_i^2.
$$

Therefore, if $\sup_n \mathbb{E}[M_n^2] = \sum_{n=1}^\infty a_n^2 < \infty$, by L^2 Martingale Convergence Theorem, $(M_n)_n$ converges almost surely and in L^2.

As for the converse direction, let us suppose $M_n \to M$ in L^2. Recall that for a square-integrable martingale and any $n \geqslant m \geqslant 0$,

$$
\mathbb{E}[M_n^2] - \mathbb{E}[M_m^2] = \mathbb{E}[(M_n - M_m)^2] \leqslant 2\mathbb{E}[(M_n - M)^2] + 2\mathbb{E}[(M_m - M)^2]
$$

since $M_n - M_m = (M_n - M) - (M_m - M)$. Hence, if $(M_n)_n$ is a convergent sequence in L^2, $(\mathbb{E}[M_n^2])_n$ is a Cauchy-sequence in $\mathbb{R}$ (follows from the last equality). Thus, uniformly bounded. But then $\sum_{n=1}^\infty a_n^2 = \sup_n \mathbb{E}[M_n^2] < \infty$.