Solution to Problem Set 5 March 18, 2025

Exercise 1.

The goal is to construct the limiting random variable. We will use the fact that $\mathbb{R}$ is a complete space, and therefore we will prove that

$$\mathbb{P}\{\omega \in \Omega : \text{ the sequence } (S_n(\omega))_{n\in\mathbb{N}} \text{ is a Cauchy sequence}\} = 1.$$

Define the following events:

$$\begin{aligned}
A(\varepsilon, n) &:= \{\omega \in \Omega : \sup_{i \geqslant n} |S_i(\omega) - S_n(\omega)| > \varepsilon\}, \\
B(\varepsilon, n) &:= \{\omega \in \Omega : \forall\, m, p \geqslant n,\, |S_m(\omega) - S_p(\omega)| \leqslant \varepsilon\}, \\
B(\varepsilon) &:= \bigcup_{n\in\mathbb{N}} B(\varepsilon, n), \\
B &:= \bigcap_{\varepsilon\in\mathbb{Q}^+} B(\varepsilon).
\end{aligned}$$

By hypothesis, $\lim_{n\to+\infty} \mathbb{P}(A(\varepsilon, n)) = 0$. Moreover, we observe that $B = \{\omega \in \Omega :$ the sequence $(S_n(\omega))_{n\in\mathbb{N}}$ is a Cauchy sequence$\}$. Indeed,

$$\begin{aligned}
\omega \in B &\Leftrightarrow \forall\, \varepsilon \in \mathbb{Q}^+,\, \omega \in B(\varepsilon) \\
&\Leftrightarrow \forall\, \varepsilon \in \mathbb{Q}^+,\, \exists\, n \in \mathbb{N} : \omega \in B(\varepsilon, n) \\
&\Leftrightarrow \forall\, \varepsilon \in \mathbb{Q}^+,\, \exists\, n \in \mathbb{N},\, \forall\, m, p \geqslant n,\, |S_m(\omega) - S_p(\omega)| \leqslant \varepsilon \\
&\Leftrightarrow (S_n(\omega))_{n\in\mathbb{N}} \text{ is a Cauchy sequence.}
\end{aligned}$$

It is thus sufficient to show that $\mathbb{P}(B) = 1$. First, we show that $\lim_{n\to+\infty} \mathbb{P}(B(\varepsilon, n)) = 1$. To do so, we show the inclusion

$$A\left(\frac{\varepsilon}{2}, n\right)^c \subseteq B(\varepsilon, n).$$

Indeed, it is easy to see that

$$A(\varepsilon, n)^c = \{\omega \in \Omega : \forall i \geqslant n,\, |S_i(\omega) - S_n(\omega)| \leqslant \varepsilon\}.$$

Consequently, if $\omega \in A(\varepsilon/2, n)^c$, then for all $i \geqslant n$, $|S_i(\omega) - S_n(\omega)| \leqslant \varepsilon/2$. For all $m, p \geqslant n$, we then have

$$\begin{aligned}
|S_m(\omega) - S_p(\omega)| &\leqslant |S_m(\omega) - S_n(\omega)| + |S_n(\omega) - S_p(\omega)| \\
&\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

and $\omega \in B(\varepsilon, n)$. Thus,

$$\mathbb{P}\left(A\left(\frac{\varepsilon}{2}, n\right)^c\right) \leqslant \mathbb{P}(B(\varepsilon, n)),$$

and by hypothesis, we can write $\lim_{n\to+\infty} \mathbb{P}(B(\varepsilon, n)) = 1$. Next, it is clear that for all $n \in \mathbb{N}$, we have $B(\varepsilon, n) \subset B(\varepsilon, n+1)$. By probability continuity, we thus have

$$1 = \lim_{n\to+\infty} \mathbb{P}(B(\varepsilon, n)) = \mathbb{P}(B(\varepsilon)).$$

Therefore, for all $\varepsilon \in \mathbb{Q}^+$, $\mathbb{P}(B(\varepsilon)^c) = 0$ and by σ-subadditivity

$$\mathbb{P}\left(\bigcup_{\varepsilon\in\mathbb{Q}^+} B(\varepsilon)^c\right) \leqslant \sum_{\varepsilon\in\mathbb{Q}^+} \mathbb{P}(B(\varepsilon)^c) = 0.$$

This means that

$$\mathbb{P}(B) = \mathbb{P}\left(\bigcap_{\varepsilon\in\mathbb{Q}^+} B(\varepsilon)\right) = 1.$$

Thus, we now define

$$S(\omega) = \begin{cases} \lim_{n\to+\infty} S_n(\omega) & \text{if } \omega \in B, \\ 0 & \text{otherwise,} \end{cases}$$

to obtain $\lim_{n\to+\infty} S_n = S$ a.s.

Exercise 2.

(a) Since f is bounded, there exists $M \in \mathbb{R}$ such that for all $i \in \mathbb{N}$, $|f(i)| \leqslant M$. Then, on the one hand, $\mathbb{E}[|Y_n|] = \mathbb{E}[|f(X_n)|] \leqslant M$. On the other hand, for all $i_1, \dots, i_n$, using the definition of Y_n, the definition of conditional expectation (discrete case), Markov property, and the hypothesis, we obtain

$$\begin{aligned} \mathbb{E}[Y_{n+1}|X_1 = i_1, \dots, X_n = i_n] &= \mathbb{E}[f(X_{n+1})|X_1 = i_1, \dots, X_n = i_n] \\ &= \sum_i f(i)\mathbb{P}(X_{n+1} = i|X_1, \dots, X_n = i_n) \\ &= \sum_i f(i)\mathbb{P}(X_{n+1} = i|X_n = i_n) \\ &= \sum_i f(i)p_{i_n,i} \\ &= f(i_n), \end{aligned}$$

for all $n \in \mathbb{N}$, and thus $\mathbb{E}[Y_{n+1}|X_1, \dots, X_n] = f(X_n) = Y_n$. Hence, (Y_n) is indeed a martingale relative to (X_n).

(b) Using the total probability formula and the hypothesis,

$$\begin{aligned} &\mathbb{P}(X_n = j \text{ for infinitely many } n) \\ &\quad= \sum_i \mathbb{P}(X_n = j \text{ for infinitely many } n|X_1 = i)\mathbb{P}(X_1 = i). \\ &\quad= \sum_i \mathbb{P}(X_1 = i) = 1. \end{aligned}$$

Since $\{X_n = j\} \subset \{Y_n = f(j)\}$, we have

$$\mathbb{P}\{Y_n = f(j) \text{ for infinitely many } n\} = 1. \tag{1}$$

Since $\mathbb{E}[Y_n^2] \leqslant M^2$, the martingale convergence theorem applies to (Y_n). Thus, there exists a random variable Y such that $\lim_{n\to+\infty} Y_n = Y$ almost surely. From (1), $\mathbb{P}\{Y = f(j)\} = 1$ for all $j \in \mathbb{N}$, and thus $f(i) = f(j)$ for all $i, j \in \mathbb{N}$. Consequently, f is a constant function.

Exercise 3.

S_τ is a random variable since it is almost surely equal to $\sum_{n=1}^{\infty} X_n \mathbf{1}_{\tau \geqslant n}$. Since $\{\tau \geqslant n\} = \{\tau \leqslant n-1\}^c$, X_n is independent of $\mathbf{1}_{\{n \leqslant \tau\}}$. Thus we have S_τ is integrable since

$$\mathbb{E}[|S_\tau|] \leqslant \sum_{n\geqslant 1} \mathbb{E}[|X_n| \cdot \mathbf{1}_{\{n\leqslant\tau\}}] = \sum_{n\geqslant 1} \mathbb{E}[|X_n|]\mathbb{E}[\mathbf{1}_{\{n\leqslant\tau\}}] = \sum_{n} \mathbb{E}[|X_1|]\mathbb{P}[n \leqslant \tau] = \mathbb{E}|X_1|\mathbb{E}[\tau].$$

Furthermore, by the dominated convergence theorem, we have

$$\mathbb{E}[S_\tau] = \mathbb{E}[\sum_{n\geqslant 1} X_n \mathbf{1}_{\{n\leqslant\tau\}}] = \sum_{n\geqslant 1} \mathbb{E}[X_n]\mathbb{E}[\mathbf{1}_{\{n\leqslant\tau\}}] = \sum_{n} \mathbb{E}[X_1]\mathbb{P}[n \leqslant \tau] = \mathbb{E}[X_1]\mathbb{E}[\tau],$$

as $\sum_{n=1}^{m} X_n \mathbf{1}_{\{n\leqslant\tau\}}$ is dominated by $\sum_{n\geqslant 1} |X_n| \cdot \mathbf{1}_{\{n\leqslant\tau\}}$ and $\mathbb{E}[\sum_{n\geqslant 1} |X_n| \cdot \mathbf{1}_{\{n\leqslant\tau\}}] = \sum_{n\geqslant 1} \mathbb{E}[|X_n| \cdot \mathbf{1}_{\{n\leqslant\tau\}}] < \infty$.