Solution to Problem Set 4 March 11, 2025

Exercise 1.

Let us first verify that $\mathbb{E}[|S_n|] < +\infty$ for all $n \geqslant 0$. Indeed, $\mathbb{E}[|S_0|] = \mathbb{E}[|X_0|] < +\infty$. For $n \geqslant 1$, using the triangle inequality and the linearity of expectation, and considering M_i as an upper bound of f_i, we obtain

$$\begin{aligned}
\mathbb{E}[|S_n|] &= \mathbb{E}\left(\left|X_0 + \sum_{i=0}^{n-1} X_{i+1} f_i(X_0, \ldots, X_i)\right|\right) \\
&\leqslant \mathbb{E}[|X_0|] + \sum_{i=0}^{n-1} \mathbb{E}[|X_{i+1}||f_i(X_0, \ldots, X_i)|] \\
&\leqslant \mathbb{E}[|X_0|] + \sum_{i=0}^{n-1} M_i \mathbb{E}[|X_{i+1}|] \\
&< +\infty,
\end{aligned}$$

by hypothesis.

It remains to prove that $\mathbb{E}[S_{n+1}|X_0, \ldots, X_n] = S_n$. Using properties (a) and (d) (see Exercise 1, Series 1), we obtain

$$\begin{aligned}
&\mathbb{E}[S_{n+1}|X_0, \ldots, X_n] \\
&= \mathbb{E}[X_0|X_0, \ldots, X_n] + \sum_{i=0}^{n-1} \mathbb{E}[X_{i+1} f_i(X_0, \ldots, X_n)|X_0, \ldots, X_n] \\
&\quad + \mathbb{E}[X_{n+1} f_n(X_0, \ldots, X_n)|X_0, \ldots, X_n] \\
&= X_0 + \sum_{i=0}^{n-1} X_{i+1} f_i(X_0, \ldots, X_n) + f_n(X_0, \ldots, X_n)\mathbb{E}[X_{n+1}|X_0, \ldots, X_n] \\
&= S_n,
\end{aligned}$$

thanks to the hypothesis.

Exercise 2.

First, suppose such a decomposition exists. Then we must have

$$S_{n+1} - S_n = M_{n+1} - M_n + A_{n+1} - A_n.$$

Consequently, by conditioning on $X_1, \ldots, X_n$ and using properties (a) and (d) (see Exercise 1, Series 1),

$$\mathbb{E}[S_{n+1}|X_1, \ldots, X_n] - S_n = A_{n+1} - A_n.$$

Thus,

$$A_{n+1} = A_n + \mathbb{E}[S_{n+1}|X_1, \ldots, X_n] - S_n.$$

Since $A_1 = 0$, we deduce that for $n \geqslant 2$,

$$A_n = \sum_{k=1}^{n-1} (\mathbb{E}[S_{k+1}|X_1, \ldots, X_k] - S_k).$$

We can then define $M_n := S_n - A_n$. It remains to verify that the sequences (A_n) and (M_n) thus defined (uniquely) satisfy the required properties. Indeed, the uniqueness of the decomposition is given by the fact that the sequence (A_n) is uniquely defined.

First, $A_1 = 0$. Next, the sequence (A_n) is increasing since for all $n \geqslant 1$ we have $A_{n+1} - A_n = \mathbb{E}[S_{n+1}|X_1, \ldots, X_n] - S_n \geqslant 0$ because (S_n) is a submartingale. Moreover, A_{n+1} is a function of $X_1, \ldots, X_n$ by the definition of conditional expectation and since $S_k, k \leqslant n$, are also such functions.

It remains to verify that (M_n) is a martingale relative to (X_n). Note that for all $n \geqslant 1$,

$$M_{n+1} - M_n = S_{n+1} - A_{n+1} - (S_n - A_n),$$

so

$$M_{n+1} = M_n + S_{n+1} - \mathbb{E}[S_{n+1}|X_1, \ldots, X_n]. \tag{1}$$

We note that $\mathbb{E}[|S_n|] < \infty$ for all $n \geqslant 1$ by the definition of a submartingale. Then, $\mathbb{E}[|M_1|] = \mathbb{E}[|S_1|] < \infty$. We proceed by induction. Assuming $\mathbb{E}[|M_n|] < \infty$, then

$$\begin{aligned} \mathbb{E}[|M_{n+1}|] &\leqslant \mathbb{E}[|M_n|] + \mathbb{E}[|S_{n+1}|] + \mathbb{E}[|\mathbb{E}[S_{n+1}|X_1, \ldots, X_n]|] \\ &< +\infty, \end{aligned}$$

by the induction hypothesis, the triangle inequality, and the definition of conditional expectation.

Furthermore, using relation (1) and properties (a) and (d) (see Ex. 1, Series 1), we obtain

$$\begin{aligned} \mathbb{E}[M_{n+1}|X_1, \ldots, X_n] &= \mathbb{E}[M_n|X_1, \ldots, X_n] + \mathbb{E}[S_{n+1}|X_1, \ldots, X_n] \\ &\qquad - \mathbb{E}[\mathbb{E}[S_{n+1}|X_1, \ldots, X_n]|X_1, \ldots, X_n] \\ &= M_n, \end{aligned}$$

which proves that (M_n) is a martingale relative to (X_n).

Exercise 3.

Let us start by showing that T is a stopping time. The random variable T represents the number of steps before drawing a green ball. The event $\{T = n\}$ is therefore equivalent to $\{R_0 = 1, R_1 = 2, R_2 = 3, \ldots, R_{n-1} = n, R_n = n\}$ (T is the first time n at which the number of red balls in the urn is n). This equality of events shows that $\{T = n\}$ is determined by the values of $R_0, \ldots, R_n$, and hence that T is a stopping time with respect to the sequence (R_n).

Since $R_T = T$, we have

$$\mathbb{E}[S_T] = \mathbb{E}\left[\frac{R_T}{T+2}\right] = \mathbb{E}\left[\frac{T}{T+2}\right] = 1 - \mathbb{E}\left[\frac{2}{T+2}\right].$$

Next, suppose that it is possible to apply the stopping theorem to the martingale (S_n) and the stopping time T. Then,

$$\mathbb{E}[S_T] = \mathbb{E}[S_1] = \frac{1}{2},$$

which leads to

$$\frac{1}{2} = 1 - 2\mathbb{E}\left[\frac{1}{T+2}\right],$$

that is,

$$\mathbb{E}\left[\frac{1}{T+2}\right] = \frac{1}{4}.$$

Thus, it remains to justify this assumption. To do so, let us first compute $\mathbb{P}\{T > n\}$. We have

$$\begin{aligned}
\mathbb{P}\{T > n\} &= \mathbb{P}\{R_0 = 1,\, R_1 = 2, \ldots, R_n = n+1\} \\
&= \mathbb{P}\{R_0 = 1\} \cdot \mathbb{P}\{R_1 = 2 | R_0 = 1\} \cdot \ldots \cdot \mathbb{P}\{R_n = n+1 | R_0 = 1, \ldots, R_n = n\} \\
&= 1 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \ldots \cdot \frac{n}{n+1} \\
&= \frac{1}{n+1}.
\end{aligned}$$

We now successively verify that the assumptions of the stopping theorem are satisfied.

(i) By continuity of probabilities,

$$\begin{aligned}
\mathbb{P}\{T < +\infty\} &= \mathbb{P}\left(\cup_{n \geqslant 1}\{T \leqslant n\}\right) \\
&= \lim_{n \to \infty} \mathbb{P}\{T \leqslant n\} \\
&= 1 - \lim_{n \to \infty} \mathbb{P}\{T > n\} = 1.
\end{aligned}$$

(ii) $\mathbb{E}[|S_T|] \leqslant 1$ since $S_T \leqslant 1$.

(iii)

$$\begin{aligned}
\mathbb{E}[S_n | T > n]\mathbb{P}\{T > n\} &= \mathbb{E}[S_n \mathbb{1}_{\{T>n\}}] \\
&\leqslant \mathbb{E}[\mathbb{1}_{\{T>n\}}] \\
&= \mathbb{P}\{T > n\} \\
&= \frac{1}{n+1} \underset{n \to +\infty}{\longrightarrow} 0
\end{aligned}$$

And thus, we can apply the stopping theorem.

Exercise 4.

Observe that equivalently, for all $n \geqslant 0$,

$$\mathbb{P}[T > n + N |_n] \leqslant 1 - \varepsilon \quad \text{a.s.}.$$

Hence, $|_n$ is short for $|X_1, \dots, X_n$, since $\{T > n\}$ is determined by $X_1, \dots, X_n$, by definition and monotonicity of conditional expectation,

$$\mathbb{P}[T > n + N] = \mathbb{E}[\mathbb{P}[T > n + N|_n]1_{\{T>n\}}] \leqslant (1 - \varepsilon)\mathbb{E}[1_{\{T>n\}}] = (1 - \varepsilon)\mathbb{P}[T > n].$$

This holds for any $n \geqslant 0$. Therefore,

$$\mathbb{P}[T > kN] \leqslant (1 - \varepsilon)\mathbb{P}[T > (k - 1)N] \leqslant \ldots \leqslant (1 - \varepsilon)^{k-1}\mathbb{P}[T > N] \leqslant (1 - \varepsilon)^k.$$

This in turn implies that $\mathbb{P}[T < \infty] \geqslant 1 - (1 - \varepsilon)^k$ for all $k \geqslant 0$. By sending k to infinity, $\mathbb{P}[T < \infty] = 1$. Moreover, since T^p is a non-negative random variable,

$$\begin{aligned}\mathbb{E}[T^p] &= \int_0^\infty \mathbb{P}[T^p > t]dt = \int_0^\infty ps^{p-1}\mathbb{P}[T > s]ds = \sum_{k=0}^\infty \int_{kN}^{(k+1)N} ps^{p-1}\mathbb{P}[T > s]ds \\ &\leqslant \sum_{k=0}^\infty \mathbb{P}[T > kN] \int_{kN}^{(k+1)N} ps^{p-1}ds \leqslant N^p \sum_{k=0}^\infty (1 - \varepsilon)^k \big((k + 1)^p - k^p\big) < \infty.\end{aligned}$$

The latter sum is clearly convergent. Fully analogously, for $\lambda > 0$,

$$\begin{aligned}\mathbb{E}[e^{\lambda T}] &= \int_0^\infty \mathbb{P}[e^{\lambda T} > t]dt = 1 + \int_1^\infty \mathbb{P}[e^{\lambda T} > t]dt = 1 + \int_0^\infty \lambda e^{\lambda s}\mathbb{P}[T > s]ds \\ &= 1 + \sum_{k=0}^\infty \int_{kN}^{(k+1)N} \lambda e^{\lambda s}\mathbb{P}[T > s]ds \leqslant 1 + \sum_{k=0}^\infty \mathbb{P}[T > kN] \int_{kN}^{(k+1)N} \lambda e^{\lambda s}ds \\ &\leqslant 1 + (e^{\lambda N} - 1) \sum_{k=0}^\infty (1 - \varepsilon)^k e^{\lambda kN}.\end{aligned}$$

The latter series is convergent if $(1-\varepsilon)e^{\lambda N} < 1$, which is the case when $\lambda > 0$ is sufficiently small.

Exercise 5.

Let $n \in \mathbb{N}$. Consider $A := \{\mathbb{E}[X_{n+1}|_n] > X_n\}$, which is determined by $X_1, \dots, X_n$. Again, $|_n$ is short for $|X_1, \dots, X_n$. Define $\tau = (n + 1)\mathbb{1}_A + n\mathbb{1}_{A^c} \leqslant n + 1$. It is clearly a bounded stopping time. Thus, on the one hand,

$$\mathbb{E}[X_0] = \mathbb{E}[X_\tau] = \mathbb{E}[X_{n+1}1_A] + \mathbb{E}[X_n 1_{A^c}].$$

On the other hand, since n is a bounded stopping time, $\mathbb{E}[X_0] = \mathbb{E}[X_n]$. Combined, we get that if $\mathbb{P}[A] > 0$,

$$\mathbb{E}[X_{n+1}1_A] = \mathbb{E}[X_n 1_A] < \mathbb{E}[\mathbb{E}[X_{n+1}|_n]1_A] = \mathbb{E}[X_{n+1}1_A],$$

which leads to a contradiction. Hence, $\mathbb{E}[X_{n+1}|_n] \leqslant X_n$ almost surely. Fully analogously, we can obtain the reverse inequality, and so $\mathbb{E}[X_{n+1}|_n] = X_n$ almost surely. Integrability are assumed.