Solution to Problem Set 3 March 4, 2025

Exercise 1.

Since (X, Y) and Z are independent, it follows that Y and Z are independent. We distinguish between the discrete and continuous cases.

Discrete case: for all y and z, we have

$$\begin{aligned}
\mathbb{E}[X|Y=y,\, Z=z] &= \sum_x x\,\mathbb{P}(X=x|Y=y,\, Z=z) \\
&= \sum_x x\,\frac{\mathbb{P}(X=x,\, Y=y,\, Z=z)}{\mathbb{P}(Y=y,\, Z=z)} \\
&\overset{\text{by independence}}{=} \sum_x x\,\frac{\mathbb{P}(X=x,\, Y=y)\mathbb{P}(Z=z)}{\mathbb{P}(Y=y)\mathbb{P}(Z=z)} \\
&= \sum_x x\,\mathbb{P}(X=x|Y=y) \\
&= \mathbb{E}[X|Y=y],
\end{aligned}$$

which is exactly what we wanted to show.

Continuous case: we assume that $f_{X,Y,Z}$, $f_{X,Y}$, $f_{Y,Z}$, f_Y and f_Z are the respective joint densities of the vectors (X, Y, Z), (X, Y), (Y, Z), Y, and Z. Then, for all y and z,

$$\begin{aligned}
\mathbb{E}[X|Y=y,\, Z=z] &= \int_{\mathbb{R}} x\frac{f_{X,Y,Z}(x,y,z)}{f_{Y,Z}(y,z)}\,dx \\
&\overset{\text{by independence}}{=} \int_{\mathbb{R}} x\frac{f_{X,Y}(x,y)f_Z(z)}{f_Y(y)f_Z(z)}\,dx \\
&= \int_{\mathbb{R}} x\frac{f_{X,Y}(x,y)}{f_Y(y)}\,dx \\
&= \mathbb{E}[X|Y=y],
\end{aligned}$$

which is exactly what we wanted to show.

Exercise 2.

(a) We use the definition of conditional expectation with the bounded function $h \equiv 1$. Then we use properties (d) and (a) of conditional expectation shown in Exercise 1 of Series 1. Finally, we take into account that (S_n) is a martingale relative to (X_n). We then obtain for all integers $k \leqslant l \leqslant m$,

$$\begin{aligned} \mathbb{E}[(S_m - S_l)S_k] &= \mathbb{E}[\mathbb{E}[(S_m - S_l)S_k | X_k, \dots, X_1]] \\ &= \mathbb{E}[S_k \, \mathbb{E}[S_m - S_l | X_k, \dots, X_1]] \\ &= \mathbb{E}[S_k(\mathbb{E}[S_m | X_k, \dots, X_1] - \mathbb{E}[S_l | X_k, \dots, X_1])] \\ &= \mathbb{E}[S_k(S_k - S_k)] = 0. \end{aligned}$$

(b) Suppose that (S_n) is a martingale relative to (X_n). The previous point can then be applied. Let $i \neq j$. Without loss of generality, we assume that $i < j$. Let $k = i$, $l = j - 1$, and $m = j$. Applying point (a), we obtain

$$\begin{aligned} 0 = \mathbb{E}[(S_m - S_l)S_k] &= \mathbb{E}[(S_j - S_{j-1})S_i] \\ &= \mathbb{E}[X_j S_i]. \end{aligned} \tag{1}$$

Since the previous equality holds for i and $i - 1$ ($i \geqslant 2$), $\mathbb{E}[X_j S_i] - \mathbb{E}[X_j S_{i-1}] = 0$. By linearity, $0 = \mathbb{E}[X_j(S_i - S_{i-1})] = \mathbb{E}[X_i X_j]$. If $i = 1$, we use equality (1) directly to obtain $\mathbb{E}[X_j X_1] = 0$.

Exercise 3.

(a) The random variable T takes values in $\{1, \dots, n\}$. Therefore, the family $\{T = 1\}$, …, $\{T = n\}$ must be a partition of Ω. This can be expressed in two ways for the sequence $B_1, \dots, B_n$:

1. *Geometric condition, valid for any sequence* $(Y_1, \dots, Y_n)$ *of random variables:* let $A_k = B_k \times \mathbb{R}^{n-k}$. Then the family $A_1, \dots, A_n$ must be a partition of $\mathbb{R}^n$. We then have

$$\{(Y_1, \dots, Y_n) \in A_k\} = \{(Y_1, \dots, Y_k) \in B_k\} = \{T = k\}.$$

2. *Probabilistic condition, valid for a given sequence* $Y_1, \dots, Y_n$ *of random variables:* let $A_k = B_k \times \mathbb{R}^{n-k}$. Then, for all $k \neq j$,
$\mathbb{P}\{(Y_1, \dots, Y_n) \in A_k \cap A_j\} = 0$ and $\mathbb{P}\{(Y_1, \dots, Y_n) \in \bigcup_{k=1}^n A_k\} = 1$.

(b) We can express the event $\{T = n\}$ as follows:

$$\begin{aligned} \{T = n\} &= \{S_n \in [1, 2) \text{ and } \; S_k \notin [1, 2), \forall \, k < n\} \\ &= \{(X_1, \dots, X_n) \in B_n\} \end{aligned}$$

where $B_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \; x_1 + \dots + x_k \notin [1, 2), \forall \, k < n \text{ and } x_1 + \dots + x_n \in [1, 2)\}$.

Exercise 4.

The sum of two stopping times is again a stopping time since

$$\{\tau_1 + \tau_2 = n\} = \bigcup_{0 \leqslant k \leqslant n} \{\tau_1 = k\} \cap \{\tau_2 = n - k\}.$$

Note that for any $k \leqslant n$, there exist B_k^1 and B_k^2 such that $\{\tau_1 = k\} = \{(Y_1, Y_2, \ldots, Y_n) \in B_k^1$ and $\{\tau_2 = k\} = \{(Y_1, Y_2, \ldots, Y_n) \in B_k^2\}$ as τ_1 and τ_2 are stopping times. Thus we have that

$$\{\tau_1+\tau_2 = n\} = \bigcup_{0 \leqslant k \leqslant n} \{(Y_1, Y_2, \ldots, Y_n) \in B_k^1 \cap B_{n-k}^2\} = \{(Y_1, Y_2, \ldots, Y_n) \in \bigcup_{0 \leqslant k \leqslant n} B_k^1 \cap B_{n-k}^2\}, \tag{2}$$

which means that $\tau_1 + \tau_2$ is a stopping time.

Exercise 5.

Define $(\Omega, \mathcal{F}, \mathbb{P})$ to be a product space of $(\Omega_i, \mathcal{P}(\Omega_i), \mathrm{Unif}(\Omega_i))$ with $\Omega_i = \{1, \ldots, 6\}^2$ for all $i \in \mathbb{N}$. For all i, let $(X_i, Y_i) : \Omega \to \mathbb{R}^2$ be a projection on the i-th coordinate, i.e. $(X_i, Y_i)(\omega) = \omega_i = (\omega_i^1, \omega_i^2) \in \{1, \ldots, 6\}^2$. Set $S_i := X_i + Y_i$, which clearly a random variable. In this model S_i clearly describes the sum of the two dice of the i-th roll. We are interested in terminating the game once S_i is even, hence, define $\tau := \{i \geqslant 1 : S_i \mod 2 \equiv 0\}$. Indeed, $\{\tau = n\} = \{S_1, \ldots, S_{n-1} \mod 2 \equiv 1, S_n \mod 2 \equiv 0\}$ and

$$\mathbb{P}[\tau > n] = \mathbb{P}\left[\cap_{i \leqslant n}\{S_i \mod 2 \equiv 1\}\right] \overset{\text{iid}}{=} \prod_{i \leqslant n} \frac{1}{2} \xrightarrow{n \to \infty} 0.$$

Here $\mathbb{P}\left[\{S_i \mod 2 \equiv 1\}\right] = \frac{1}{2}$ because the second dice has half the probability of being even or odd after the first dice is given. For the first roll S_1, one may check that its distribution is as below:

2	3	4	5	6	7	8	9	10	11	12
$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Formally, it is supported on $\{2, 3, \ldots, 12\}$ with

$$\mathbb{P}[S_1 = k] = \frac{1}{36}\big((k-1)1_{k \leqslant 7} + (13-k)1_{k>7}\big)$$

for $k = 2, 3, \ldots, 12$. Hence, for $k \in \{2, 4, \ldots, 12\}$,

$$\begin{aligned}
\mathbb{P}[S_\tau = k] &= \sum_{i \in \mathbb{N}} \mathbb{P}[S_i = k, \tau = i] = \sum_{i \in \mathbb{N}} \mathbb{P}[S_i = k; S_1, \ldots S_{i-1} \mod 2 \equiv 1] \\
&\overset{\text{iid}}{=} \sum_{i \in \mathbb{N}} \mathbb{P}[S_i = k] \prod_{j<i} \frac{1}{2} = \mathbb{P}[S_1 = k] \sum_{i \in \mathbb{N}} \left(\frac{1}{2}\right)^{i-1} = 2\mathbb{P}[S_1 = k] \qquad (3) \\
&= \frac{1}{18}\big((k-1)1_{k \leqslant 7} + (13-k)1_{k>7}\big).
\end{aligned}$$

Note that the law of S_τ is obtained from the law of S_1 by deleting the mass from odd numbers and doubling the mass of even numbers.