Solution to Problem Set 13 May 20, 2025

Exercise 1.

Let $(B_t^{(2)}, t \in [0,1])$ be a process identical to $(B_t^{(1)}, t \in [0,1])$ and independent. Then define $B_t^{(3)} = tB^{(2)}_{\frac{1}{t}}$, for all $t \geqslant 1$. Note that $(B_t^{(3)}, t \geqslant 1)$ is independent of $(B_t^{(1)}, t \in [0,1])$ and that it satisfies properties (a) and (b) of the definition of Brownian motion. Now, define

$$B_t = \begin{cases} B_t^{(1)} & \text{if } t \in [0,1], \\ B_1^{(1)} + B_t^{(3)} - B_1^{(3)} & \text{if } t \geqslant 1. \end{cases}$$

We already note that the trajectories are continuous by construction. It remains to show that $(B_t, t \geqslant 0)$ satisfies properties (a) and (b) of the definition of Brownian motion. We begin by verifying (a). There are three cases.

- **Case 1:** $t + s < 1$. In this case,

$$B_{t+s} - B_s = B_{t+s}^{(1)} - B_s^{(1)} \rightsquigarrow N(0,t),$$

since $(B_t^{(1)}, t \geqslant 0)$ satisfies (a) of the definition of Brownian motion.

- **Case 2:** $s > 1$. In this case,

$$\begin{aligned} B_{t+s} - B_s &= B_1^{(1)} + B_{t+s}^{(3)} - B_1^{(3)} - B_1^{(1)} - B_s^{(3)} + B_1^{(1)} \\ &= B_{t+s}^{(3)} - B_s^{(3)} \rightsquigarrow N(0,t), \end{aligned}$$

since $(B_t^{(3)}, t \geqslant 0)$ satisfies (a) of the definition of Brownian motion.

- **Case 3:** $s < 1 < t + s$. In this case,

$$\begin{aligned} B_{t+s} - B_s &= B_1^{(1)} + B_{t+s}^{(3)} - B_1^{(3)} - B_s^{(1)} \\ &= \big(B_1^{(1)} - B_s^{(1)}\big) + \big(B_{t+s}^{(3)} - B_1^{(3)}\big). \end{aligned}$$

But the two random variables $B_1^{(1)} - B_s^{(1)}$ and $B_{t+s}^{(3)} - B_1^{(3)}$ are Gaussian, centered and independent. Their sum is a centered Gaussian random variable whose variance is the sum of the variances. Thus, $B_{t+s} - B_s \rightsquigarrow N(0,t)$.

This shows that the process $(B_t, t \geqslant 0)$ satisfies property (a) of the definition of Brownian motion. It remains to show that it satisfies property (b). Let $t_1 < t_2 \leqslant t_3 < t_4$ be positive real numbers. Again, there are several cases to discuss.

- **Case 1:** $t_4 < 1$. Then the increments considered are those of the process $(B_t^{(1)}, t \in [0,1])$ and they are independent because the increments of $B^{(1)}$ are independent.
- **Case 2:** $t_1 \leqslant 1 < t_2$. In this case, the increments are

$$\begin{aligned} B_{t_2} - B_{t_1} &= B_1^{(1)} + B_{t_2}^{(3)} - B_1^{(3)} - B_{t_1}^{(1)}, \\ B_{t_4} - B_{t_3} &= B_1^{(1)} + B_{t_4}^{(3)} - B_1^{(3)} - B_{t_3}^{(1)} \\ &= B_{t_4}^{(3)} - B_{t_3}^{(3)}. \end{aligned}$$

And the increments are indeed independent, because $(B_t^{(1)})$ and $(B_t^{(3)})$ are independent, and $B^{(3)}$ satisfies property (b) of the definition of Brownian motion. The remaining cases $t_1 < t_2 \leqslant 1 < t_3 < t_4$, $t_1 < t_2 < t_3 \leqslant 1 < t_4$, and $t_1 \geqslant 1$ can be verified in the same way as above.

Exercise 2.

(a) Let ω be in F. Then there exists a $t_0 \in [0,1]$ such that B'_{t_0} exists and is finite. That is,

$$\lim_{t \to t_0} \frac{|B_t(\omega) - B_{t_0}(\omega)|}{|t - t_0|} = |B'_{t_0}(\omega)| < \infty.$$

Then, by the definition of the limit, one can find $N \in \mathbb{N}$ and $n \in \mathbb{N}^*$ such that

$$|B_t(\omega) - B_{t_0}(\omega)| \leqslant N|t - t_0|, \quad \text{for all } t \text{ such that } |t - t_0| \leqslant \frac{3}{n}.$$

Thus ω belongs to $G_{N,n}$, which shows the inclusion.

(b) Let $\omega \in G_{N,n}$. Then there exists a $t_0 \in [0,1]$ and an $i \in \{1, \dots, n-2\}$ such that $t_0 \in \left[\frac{i-1}{n}, \frac{i+2}{n}\right]$ and

$$|B_t(\omega) - B_{t_0}(\omega)| \leqslant N|t - t_0|, \quad \text{for all } t \in \left[\frac{i-1}{n}, \frac{i+2}{n}\right].$$

Then, by the triangle inequality, we have

$$\begin{aligned} \left|B_{\frac{i}{n}}(\omega) - B_{\frac{i-1}{n}}(\omega)\right| &\leqslant \left|B_{\frac{i}{n}}(\omega) - B_{t_0}(\omega)\right| + \left|B_{t_0}(\omega) - B_{\frac{i-1}{n}}(\omega)\right| \\ &\leqslant N\left|\frac{i}{n} - t_0\right| + N\left|\frac{i-1}{n} - t_0\right| \leqslant N\left(\frac{2}{n} + \frac{3}{n}\right). \end{aligned}$$

Similarly,

$$\begin{aligned} \left|B_{\frac{i+1}{n}}(\omega) - B_{\frac{i}{n}}(\omega)\right| &\leqslant N\left(\frac{2}{n} + \frac{2}{n}\right), \\ \left|B_{\frac{i+2}{n}}(\omega) - B_{\frac{i+1}{n}}(\omega)\right| &\leqslant N\left(\frac{2}{n} + \frac{3}{n}\right). \end{aligned}$$

Thus, $Y_{i,n}(\omega) \leqslant \frac{5N}{n}$ and hence ω belongs to $H_{N,n}$. This inclusion is thereby established.

(c) By properties (a) and (b) in the definition of Brownian motion, the collection

$$\left(\left|B_{\frac{k+j}{n}} - B_{\frac{k+j-1}{n}}\right|, \, j = 0, 1, 2\right)$$

is i.i.d. with law $\mathcal{N}(0, \frac{1}{n})$. The maximum of these three increments will be less than $\frac{5N}{n}$ if and only if all three increments are less than $\frac{5N}{n}$. Hence,

$$\mathbb{P}\left\{Y_{k,n} \leqslant \frac{5N}{n}\right\} = \left(\mathbb{P}\left\{\left|B_{\frac{1}{n}}\right| \leqslant \frac{5N}{n}\right\}\right)^3.$$

Moreover, using the scaling property, we have

$$\mathbb{P}\left\{Y_{k,n} \leqslant \frac{5N}{n}\right\} = \left(\mathbb{P}\left\{|B_1| \leqslant \frac{5N}{\sqrt{n}}\right\}\right)^3.$$

Then,

$$\begin{aligned}
\mathbb{P}(H_{N,n}) &\leqslant n\Big(\mathbb{P}\left\{|B_1| \leqslant \frac{5N}{\sqrt{n}}\right\}\Big)^3 = n\left(2\int_0^{\frac{5N}{\sqrt{n}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx\right)^3 \\
&\leqslant n\left(2\int_0^{\frac{5N}{\sqrt{n}}} \frac{1}{\sqrt{2\pi}}\, dx\right)^3 = \left(\frac{10N}{\sqrt{2\pi}}\right)^3 \frac{1}{\sqrt{n}} \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Finally, note that $G_{N,n} \subset G_{N,n+1}$. Then, the inclusions we have established in parts (a) and (b) allow us to conclude. Indeed, we have

$$\mathbb{P}(G_{N,n}) \leqslant \lim_{m\to\infty} \mathbb{P}(G_{N,m}) \leqslant \lim_{m\to\infty} \mathbb{P}(H_{N,m}) = 0.$$

This shows that $\mathbb{P}(G_{N,n}) = 0$ for all $n \in \mathbb{N}^*$ and for all $N \in \mathbb{N}$, and hence $\mathbb{P}(F) = 0$.

Exercise 3.

1. The continuity of B implies that $\mathbb{Z}$ is almost surely closed. So we only need to check that it almost surely has no isolated points.
   We define $\tau_q := \inf\{t \geqslant q : B_t = 0\}$, which is almost surely finite for all q because $\mathcal{Z}$ is unbounded. Since $B_{\tau_q} = 0$, by the strong Markov property, we know that $B_{t+\tau_q}$ is again a Brownian motion. Furthermore, for any Brownian motion $\tilde{B}$, we almost surely have $\inf\{t > 0 : \tilde{B}_t = 0\} = 0$. Thus, almost surely, for every $q \in \mathbb{Q}^+$, τ_q is not an isolated point.

   Finally, if $t \in \mathcal{Z} \setminus \{\tau_q : q \in \mathbb{Q}^+\}$, we consider a sequence of rational numbers $q_n \uparrow t$, and we observe that $q_n \leqslant \tau_{q_n} < t$, hence t is not isolated.

2. $$\mathbb{E}\left[\int_0^\infty \mathbf{1}_{\mathcal{Z}}(t)\, dt\right] = \int_0^\infty \mathbb{E}\left[\mathbf{1}_{\{B_t=0\}}\right]\, dt = \int_0^\infty \mathbb{P}[B_t = 0]\, dt = 0.$$