Solution to Problem Set 12 May 13, 2025

Exercise 1.

(a) We observe that $\frac{b+B_t}{1+t} \geqslant a$ if and only if $B_t - at \geqslant a - b$, so that

$$\mathbb{P}\left\{\sup_{t\geqslant 0} \frac{b+B_t}{1+t} \geqslant a\right\} = \mathbb{P}\left\{\sup_{t\geqslant 0}(B_t - at) \geqslant a-b\right\}.$$

Since the process $(B_t - at,\, t \geqslant 0)$ is a Brownian motion with negative drift $(-a)$, this probability equals $e^{-2a(a-b)}$ by a result from the course.

(b) Let $(\widehat{B}_t,\, t \geqslant 0)$ be a Brownian motion and let $(B'_t,\, t \geqslant 0)$ and $(\widetilde{B}_t,\, t \geqslant 0)$ be standard Brownian motions. Define $B^*_t = B_{t-1}$ for all $t \geqslant 1$. Then $(B^*_t,\, t \geqslant 1)$ is a Brownian motion. Hence,

$$\begin{aligned}
\mathbb{P}\left\{\sup_{t\geqslant 0} \frac{b+B_t}{1+t} \geqslant a\right\} &= \mathbb{P}\left\{\sup_{t\geqslant 0} \frac{\widehat{B}_t}{1+t} \geqslant a \,\Big|\, \widehat{B}_0 = b\right\} \\
&= \mathbb{P}\left\{\sup_{t\geqslant 0} \frac{B^*_{1+t}}{1+t} \geqslant a \,\Big|\, B^*_1 = b\right\} \\
&= \mathbb{P}\left\{\sup_{t\geqslant 1} \frac{B'_t}{t} \geqslant a \,\Big|\, B'_1 = b\right\}.
\end{aligned}$$

Therefore, using the fact that $(tB'_{1/t},\, t \geqslant 0)$ is a standard Brownian motion, it follows that

$$\begin{aligned}
\mathbb{P}\left\{\sup_{t\geqslant 1} \frac{B'_t}{t} \geqslant a \,\Big|\, B'_1 = b\right\} &= \mathbb{P}\left\{\sup_{t\geqslant 1} \widetilde{B}_{1/t} \geqslant a \,\Big|\, \widetilde{B}_1 = b\right\} \\
&= \mathbb{P}_0\left\{\sup_{0\leqslant t\leqslant 1} \widetilde{B}_t \geqslant a \,\Big|\, \widetilde{B}_1 = b\right\}.
\end{aligned}$$

Exercise 2.

Let $\widetilde{B}_t = tB_{\frac{1}{t}}$. Then $(\widetilde{B}_t,\, t \geqslant 0)$ is a standard Brownian motion (by setting, in addition, $\widetilde{B}_0 = 0$). Using a result from the course, we have

$$\mathbb{P}_0\left\{\bigcap_{n\in\mathbb{N}^*}\left\{\exists t \in \left(0, \frac{1}{n}\right) : B_t = 0\right\}\right\} = 1.$$

By letting $s = \frac{1}{t}$, this relation is equivalent to

$$\mathbb{P}_0\left\{\bigcap_{n\in\mathbb{N}^*}\{\exists s \geqslant n : B_{1/s} = 0\}\right\} = 1,$$

which is itself equivalent (renaming $s = t$) to

$$\mathbb{P}_0 \left\{ \bigcap_{n \in \mathbb{N}^*} \{\exists t \geqslant n : tB_{1/t} = 0\} \right\} = 1.$$

And by using the process $(\widetilde{B}_t,\, t \geqslant 0)$, it follows that

$$\mathbb{P}_0 \left\{ \bigcap_{n \in \mathbb{N}^*} \{\exists t \geqslant n : \widetilde{B}_t = 0\} \right\} = 1.$$

Exercise 3.

First, we will establish a result that will be needed later. We know that the sum of two independent Gaussians is Gaussian. A more general result tells us that any linear combination of a Gaussian vector is a Gaussian vector. The case of interest is the vector

$$\vec{X} = (B_{t_1}, B_{t_2}, \ldots, B_{t_n})^T,$$

where $(B_t,\, t \geqslant 0)$ is a standard Brownian motion and $0 \leqslant t_1 < \cdots < t_n < t$. We know from the course that the vector

$$\vec{Y} = (B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})^T$$

follows a law $\mathcal{N}_n(\vec{0}, AA^T)$, where

$$A = \begin{pmatrix} \sqrt{t_1} & & & 0 \\ & \sqrt{t_2 - t_1} & & \\ & & \ddots & \\ 0 & & & \sqrt{t_n - t_{n-1}} \end{pmatrix}.$$

Moreover, we can write $\vec{X} = S\vec{Y}$ with

$$S = \begin{pmatrix} 1 & & & 0 \\ 1 & 1 & & \\ & \ddots & \ddots & \\ 1 & & 1 & 1 \end{pmatrix}.$$

By setting $\Sigma = SA$, we find that $\vec{X}$ follows a multivariate normal law $\mathcal{N}_n(\vec{0}, \Sigma\Sigma^T)$. One can check that the joint density of the $X_i = B_{t_i}$ given in the course is indeed

$$f_{\vec{X}}(x_1, \ldots, x_n) = \left(\frac{1}{2\pi}\right)^{n/2} \frac{1}{|\det \Sigma|} \exp\left\{ -\frac{1}{2} \vec{x}^T \left(\Sigma\Sigma^T\right)^{-1} \vec{x} \right\}.$$

Thus, any linear combination of the coordinates of $\vec{X}$ will follow a multivariate Gaussian law. We will use this in part (b).

We now return to the exercise at hand.

(a) Using Fubini's theorem, we find that

$$\begin{aligned}
\mathbb{E}(C_k) &= \frac{2}{\pi}\mathbb{E}\Big(\int_0^\pi \Big(B_t - \frac{t}{\pi}B_\pi\Big)\sin(kt)\,dt\Big) = \frac{2}{\pi}\int_0^\pi \mathbb{E}\Big(B_t - \frac{t}{\pi}B_\pi\Big)\sin(kt)\,dt = 0,\\
\mathbb{E}(C_kC_l) &= \Big(\frac{2}{\pi}\Big)^2\,\mathbb{E}\Big(\int_0^\pi \Big(B_t - \frac{t}{\pi}B_\pi\Big)\sin(kt)\,dt \int_0^\pi \Big(B_s - \frac{s}{\pi}B_\pi\Big)\sin(ls)\,ds\Big)\\
&= \Big(\frac{2}{\pi}\Big)^2 \int_0^\pi dt \int_0^\pi ds\ \sin(kt)\sin(ls)\,\mathbb{E}\Big((B_t - \frac{t}{\pi}B_\pi)(B_s - \frac{s}{\pi}B_\pi)\Big)\\
&= \Big(\frac{2}{\pi}\Big)^2 \int_0^\pi dt \int_0^\pi ds\ \sin(kt)\sin(ls)\,\Big(s\wedge t\Big(1-\frac{s\vee t}{\pi}\Big)\Big)\\
&= \Big(\frac{2}{\pi}\Big)^2 \int_0^\pi dt\Big(\int_0^t ds\ \sin(kt)\sin(ls)\,s\Big(1-\frac{t}{\pi}\Big) + \int_t^\pi ds\ \sin(kt)\sin(ls)\,t\Big(1-\frac{s}{\pi}\Big)\Big)\\
&= \Big(\frac{2}{\pi}\Big)^2 \int_0^\pi dt\Big((1-\frac{t}{\pi})\sin(kt)\Big[\frac{\sin(ls)}{l^2} - \frac{\cos(ls)s}{l}\Big]_0^t\\
&\qquad + \sin(kt)t\Big[-\frac{\cos(ls)(1-\frac{s}{\pi})}{l} - \frac{\sin(ls)}{l^2\pi}\Big]_t^\pi\Big)\\
&= \Big(\frac{2}{\pi}\Big)^2 \int_0^\pi dt\Big((1-\frac{t}{\pi})\sin(kt)\frac{\sin(lt)}{l^2} - (1-\frac{t}{\pi})\sin(kt)\frac{\cos(lt)t}{l}\Big)\\
&\qquad + \Big(\frac{2}{\pi}\Big)^2 \int_0^\pi dt\Big(\sin(kt)t\frac{\cos(lt)(1-\frac{t}{\pi})}{l} + \sin(kt)t\frac{\sin(lt)}{l^2\pi}\Big)\\
&= \Big(\frac{2}{\pi}\Big)^2 \int_0^\pi \frac{1}{l^2}\sin(kt)\sin(lt)\,dt\\
&= \begin{cases} 0, & \text{if } k\neq l,\\ \frac{2}{\pi l^2}, & \text{if } k = l.\end{cases}
\end{aligned}$$

(b) To determine the law of C_k, we write the integral as a Riemann sum. Define

$$C_k^{(n)} = \frac{2}{\pi}\sum_{i=1}^n \frac{\pi}{n}\Big(B_{\frac{i\pi}{n}} - \frac{1}{\pi}\frac{i\pi}{n}B_\pi\Big)\sin\Big(k\frac{i\pi}{n}\Big).$$

By the continuity of B_t, $C_k^{(n)}$ converges almost surely to C_k. It is then necessary to find the law of this limit. Since $C_k^{(n)}$ can be written as a linear combination of the coordinates of the vector $\vec{X} = (B_{\frac{\pi}{n}}, B_{\frac{2\pi}{n}}, \ldots, B_\pi)^T$, we know from part (a) above that $C_k^{(n)}$ follows a normal law $\mathcal{N}(0,\sigma_n^2)$. That is, its characteristic function is given by

$$\phi_n(t) = \mathbb{E}\Big(e^{itC_k^{(n)}}\Big) = e^{-\sigma_n^2\frac{t^2}{2}}, \quad \forall\, t\in\mathbb{R}.$$

Since almost sure convergence implies convergence in law, the characteristic function of $C_k^{(n)}$ must converge. Thus $\sigma_n \to \sigma$ and the limit C_k follows a normal law $\mathcal{N}(0,\sigma^2)$, where $\sigma^2 = \frac{2}{\pi k^2}$ by part (a).

(c) By part (b), we know that $(C_{j_1}, \ldots, C_{j_n})$, for any $j_1, \ldots, j_n \in \mathbb{N}$, is the limit of a multivariate normal vector. Therefore, this vector is also multivariate normal. Moreover, as shown in part (a), the C_{j_i} are uncorrelated, which implies their independence. Furthermore, we check that

$$\begin{aligned}
\mathbb{E}(C_k B_\pi) &= \left(\frac{2}{\pi}\right)^2 \mathbb{E}\left(\int_0^\pi \left(B_t - \frac{t}{\pi} B_\pi\right) \sin(kt)\, dt \cdot B_\pi\right) \\
&= \left(\frac{2}{\pi}\right)^2 \int_0^\pi ds\ \sin(kt)\, \mathbb{E}\Big((B_t - \frac{t}{\pi} B_\pi) B_\pi\Big) \\
&= 0.
\end{aligned}$$

Thus, they are also independent of B_π.

(d) Note that $B_t - \frac{t}{\pi} B_\pi$ is a.s. a continuous function and the value of it at 0 and π are both 0, we have that $B_t - \frac{t}{\pi} B_\pi \in L^2[0, \pi]$. Thus, by the theory of Fourier series, we know that C_k are simply its Fourier coefficients, which means that

$$B_t - \frac{t}{\pi} B_\pi = \sum_{k=1}^{\infty} C_k \sin(kt). \tag{1}$$

By (b) and (c), we can define $Y_0 := \frac{1}{\sqrt{\pi}} B_\pi$ and $Y_k := k\sqrt{\frac{\pi}{2}} C_k$ for $k \geqslant 1$ so that $(Y_k)_{k \in \mathbb{N}}$ are i.i.d $N(0, 1)$ random variables. And we will have that

$$B_t = \frac{t}{\sqrt{\pi}} Y_0 + \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} \frac{\sin kt}{k} Y_k. \tag{2}$$

This implies that (X_t) and (B_t) have the same law.