Solution to Problem Set 11 May 6, 2025

Exercise 1.

The idea is the same as in Exercise 3 of Problem Set 10. Since Brownian motion is a time-homogeneous Markov process, we can write

$$\begin{aligned}\frac{\mathbb{E}[f(Z_{t+h}) - f(x) \mid Z_t = x]}{h} &= \frac{\mathbb{E}[f(Z_h) - f(x) \mid Z_0 = x]}{h} \\ &= \frac{\mathbb{E}[f(B_h + \mu h + x) - f(x)]}{h}.\end{aligned}$$

Using a second order Taylor expansion in the neighborhood of $B_0 = Z_0 = x$, there exists a random variable $\theta \in (0,1)$ such that

$$\begin{aligned}&\frac{\mathbb{E}[f(Z_{t+h}) - f(x) \mid Z_t = x]}{h} \\ &= \mathbb{E}\left[f'(x)\frac{B_h + \mu h}{h} + f''(x + \theta(B_h + \mu h))\frac{(B_h + \mu h)^2}{2h}\right] \\ &= \mu f'(x) + \frac{f''(x)}{2}\frac{1}{h}\mathbb{E}[B_h^2 + 2\mu B_h + \mu^2 h^2] \\ &\quad + \mathbb{E}\left[\Big(f''(x + \theta(B_h + \mu h)) - f''(x)\Big)\frac{(B_h + \mu h)^2}{2h}\right] \\ &= \mu f'(x) + \frac{f''(x)}{2}(1 + \mu^2 h) \\ &\quad + \mathbb{E}\left[\Big(f''(x + \theta(B_h + \mu h)) - f''(x)\Big)\frac{(B_h + \mu h)^2}{2h}\right]. \end{aligned} \tag{1}$$

One can show, in the same manner as in Exercise 3 of Problem Set 10, that the last term on the right-hand side of (1) tends to 0 as $h \to 0$. Taking the limit as h tends to 0 in the first two terms of (1), we obtain the desired result.

Exercise 2.

(a) It was shown in class that $\mathbb{P}\{M_t \geqslant m,\ B_t \leqslant x\} = \mathbb{P}\{B_t \geqslant 2m - x\}$. Thus,

$$\begin{aligned}f_{M_t, B_t}(m, x) &= -\frac{\partial}{\partial m}\frac{\partial}{\partial x}\left(1 - \int_{-\infty}^{2m-x} p(y,t)\,dy\right) \\ &= -\frac{\partial}{\partial m}p(2m - x, t), \\ &= -2\frac{\partial p(2m - x, t)}{\partial u},\end{aligned} \tag{2}$$

or, more explicitly,

$$f_{M_t,\,B_t}(m,\,x) = \frac{2m-x}{t}\frac{2}{\sqrt{2\pi t}}\exp\left(-\frac{(2m-x)^2}{2t}\right).$$

It suffices to perform the change of variables $(x, y) \mapsto (x, x-y)$, with Jacobian matrix

$$J = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix},$$

with $|\det J| = 1$, to obtain

$$f_{M_t,\,M_t-B_t}(m, y) = f_{M_t,\,B_t}(m, m-y),$$

or more explicitly,

$$f_{M_t,\,M_t-B_t}(m, y) = \frac{m+y}{t}\frac{2}{\sqrt{2\pi t}}\exp\left(-\frac{(m+y)^2}{2t}\right).$$

(b) We use the joint density computed in (a) to find the marginal law. In particular, using relation (2), we have

$$\begin{aligned} f_{M_t-B_t}(y) &= \int_0^{+\infty} f_{M_t,\,M_t-B_t}(m, y)\,dm \\ &= \int_0^{+\infty} -2\frac{\partial}{\partial u}p(m+y, t)\,dm \\ &= 2p(y, t). \end{aligned}$$

Note that this is indeed a density, since the function is positive and $\int_0^{+\infty} 2p(y, t)\,dy = 1$.

(c) It suffices to use the answers from (a) and (b) to obtain

$$\begin{aligned} \mathbb{P}\{M_t > \xi \mid B_t = M_t\} &= \int_\xi^{+\infty} f_{M_t|\,B_t-M_t=0}(y)\,dy \\ &= \int_\xi^{+\infty} \frac{f_{M_t,\,M_t-B_t}(y, 0)}{f_{M_t-B_t}(0)}\,dy \\ &= \int_\xi^{+\infty} -\frac{2\,\frac{\partial p(y,t)}{\partial u}}{2p(0, t)}\,dy \\ &= \frac{p(\xi, t)}{p(0, t)} \\ &= \exp\left(-\frac{\xi^2}{2t}\right). \end{aligned}$$

Exercise 3.

Let Y be a random variable independent of $B = (B_t)_{t\in[0,1]}$ and it has a uniform distribution on $[0, 1]$. Then $(W_t)_{t\in[0,1]}$ defined as

$$W_t := B_t + \delta(t - Y) \quad \forall t \in [0, 1] \tag{3}$$

satisfies the desired property. Indeed, since $\mathbb{P}[\delta(t-Y)=0]=\mathbb{P}[Y\neq t]=1$ for any $t\in[0,1]$ (the same holds for at most countably many points simultaneously), the marginal laws of W and B coincide. It is also clear that $t\mapsto\delta(t-Y)$ is almost surely discontinuous, and so is the process W.

Exercise 4.

Let $[a,b]$ be an arbitrary non-degenerate interval. If it is an interval of monotonicity, then for any $a=t_1\leqslant t_2\leqslant\ldots\leqslant t_{n+1}=b$, all the increments $(B_{t_{k+1}}-B_{t_k})_{k=1}^n$ have to have the same sign. But by definition of the BM, these increments are independent, hence this event has probability $2*2^{-n}$. By sending n to ∞, we see that the probability of B being monotone on $[a,b]$ is zero. (One can also see this from the reflection principle.) Since the union of countable zero sets is again a zero set, we can conclude the same result simultaneously for countably many intervals. In particular,

$$\mathbb{P}\Big[\bigcup_{0<a<b\in\mathbb{Q}}\{t\in[a,b]\mapsto B_t\text{ is monotone}\}\Big]=0. \tag{4}$$

But by density of $\mathbb{Q}$ in $\mathbb{R}$, every non-degenerate interval will contain a non-degenerate interval with rational points. Therefore,

$$\mathbb{P}\Big[\bigcup_{0<a<b\in\mathbb{R}}\{t\in[a,b]\mapsto B_t\text{ is monotone}\}\Big]=0 \tag{5}$$

Exercise 5.

Recall that in the proof of Exercise 4 Sheet 10, we showed that a.s, $\sup_t B_t=+\infty$. The same argument applied to $(-B_t)_t$ implies that $\inf_t B_t=-\infty$ almost surely. This and the continuity of BM directly imply that both $T_{-a,b}$ and T_a are almost surely finite.