Solution to Problem Set 10 April 29, 2025

Exercise 1.

The reflected Brownian motion is nothing other than the process $W_t = |B_t|$, where (B_t) is a standard Brownian motion. The computation of the expectation is straightforward. We have, since the integrand is an even function,

$$\begin{aligned}
\mathbb{E}[W_t] &= \int_{-\infty}^{+\infty} |x|\, p(x,t)\, dx \\
&= 2\int_0^{+\infty} x\, \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right)\, dx \\
&= \sqrt{\frac{2}{\pi t}} \left[-t \exp\left(-\frac{x^2}{2t}\right)\right]_0^{+\infty} \\
&= \sqrt{\frac{2t}{\pi}}.
\end{aligned}$$

The computation of the variance is done via the formula $\mathrm{Var}(W_t) = \mathbb{E}[W_t^2] - \big(\mathbb{E}[W_t]\big)^2$. Thus,

$$\begin{aligned}
\mathbb{E}[W_t^2] &= \mathbb{E}[|B_t|^2] \\
&= t.
\end{aligned}$$

We deduce that the variance of W_t is $t - \frac{2t}{\pi}$.

Exercise 2.

Before finding the conditional density, we will first determine the joint density of the variables $X_t = \min_{0\leq u\leq t} B_u$ and B_t. Note that for $c < a \wedge b$,

$$\begin{aligned}
&\mathbb{P}\{B_u \neq c \text{ for } 0 \leq u \leq t,\ B_t > b \mid B_0 = a\} \\
&\quad = \mathbb{P}_a\{X_t > c,\ B_t > b\} \\
&\quad = \mathbb{P}_a\{B_t > b\} - \mathbb{P}_a\{X_t \leq c,\ B_t > b\}.
\end{aligned}$$

Using the reflection principle with respect to the level c, the second term becomes

$$\mathbb{P}_a\{X_t \leq c,\ B_t > b\} = \mathbb{P}_a\{B_t < -b + 2c\}.$$

By a translation property of B_t and the fact that for a standard Brownian motion, $(-B_t)$

and (B_t) have the same law, we obtain that for $c < a \wedge b$,

$$\begin{aligned}
&\mathbb{P}\{B_u \neq c \text{ for } 0 \le u \le t,\ B_t > b \mid B_0 = a\} \\
&\quad = \mathbb{P}_a\{B_t > b\} - \mathbb{P}_a\{B_t < -b + 2c\} \\
&\quad = \mathbb{P}_0\{B_t > b - a\} - \mathbb{P}_0\{B_t < -a - b + 2c\} \\
&\quad = \mathbb{P}_0\{B_t > b - a\} - \mathbb{P}_0\{B_t \ge a + b - 2c\} \\
&\quad = \int_{b-a}^{a+b-2c} p(u,t)\, du.
\end{aligned}$$

Since the joint density (of X_t and B_t) is given by

$$f^a_{X_t,B_t}(c,b) = \frac{\partial}{\partial b}\frac{\partial}{\partial c}\mathbb{P}_a\{X_t > c,\ B_t > b\},$$

we find

$$\begin{aligned}
f^a_{X_t,B_t}(c,b) &= \frac{\partial}{\partial b}\frac{\partial}{\partial c}\int_{b-a}^{a+b-2c} p(u,t)\, du \\
&= -2\,\frac{\partial}{\partial b} p(a+b-2c,t).
\end{aligned}$$

Thus,

$$\begin{aligned}
f^a_{X_t|B_t=b}(c) &= \frac{f^a_{X_t,B_t}(c,b)}{f^a_{B_t}(b)} \\
&= \frac{-2\,\frac{\partial}{\partial b} p(a+b-2c,t)}{p(b-a,t)},
\end{aligned}$$

and therefore

$$\begin{aligned}
\mathbb{P}_a\{X_t > 0 \mid B_t = b\} &= \int_0^{b\wedge a} \frac{-2\,\frac{\partial}{\partial b} p(a+b-2c,t)}{p(b-a,t)}\, dc \\
&= \frac{\big[\, p(a+b-2c,t)\,\big]_{c=0}^{c=b\wedge a}}{p(a-b,t)} \\
&= 1 - \frac{p(a+b,t)}{p(a-b,t)} \\
&= 1 - e^{-\frac{2ab}{t}}.
\end{aligned}$$

Exercise 3.

We begin by noting that by the Markov property of Brownian motion, we have

$$\begin{aligned}
\mathbb{E}\big[f(B_{t+h}) \mid B_t = x\big] &= \mathbb{E}\big[f(B_h) \mid B_0 = x\big] \\
&= \mathbb{E}\big[f(x + B_h) \mid B_0 = 0\big] \\
&= \mathbb{E}_0\big[f(x + B_h)\big].
\end{aligned}$$

Thus, by performing a Taylor expansion at the point x, there exists a random variable

$$\theta \in]0,1[,$$

such that

$$\begin{aligned}
&\frac{1}{h}\Big(\mathbb{E}\big[f(B_{t+h}) \mid B_t = x\big] - f(x)\Big) \\
&= \frac{1}{h}\mathbb{E}\big[f(x+B_h) - f(x)\big] \\
&= \frac{1}{h}\mathbb{E}\Big[f(x) + B_h f'(x) + \frac{B_h^2}{2} f''(x+\theta B_h) - f(x)\Big] \\
&= \frac{1}{h} f'(x)\mathbb{E}[B_h] + \mathbb{E}\Big[\frac{\frac{B_h^2}{2} f''(x+\theta B_h)}{h}\Big] \\
&= 0 + \mathbb{E}\Big[\frac{\frac{B_h^2}{2} f''(x+\theta B_h)}{h}\Big] \\
&= \mathbb{E}\Big[\frac{B_h^2}{h}\Big]\frac{f''(x)}{2} + \mathbb{E}\Big[\frac{B_h^2}{2h}\Big(f''(x+\theta B_h) - f''(x)\Big)\Big] \\
&= \frac{f''(x)}{2} + \mathbb{E}\Big[\frac{B_h^2}{2h}\Big(f''(x+\theta B_h) - f''(x)\Big)\Big]. \qquad (1)
\end{aligned}$$

We must therefore show that

$$\lim_{h\downarrow 0} \mathbb{E}\Big[\frac{B_h^2}{2h}\Big(f''(x+\theta B_h) - f''(x)\Big)\Big] = 0.$$

Note that by the scaling invariance, we have

$$\begin{aligned}
\mathbb{E}\Big[\frac{B_h^2}{2h}\Big|f''(x+\theta B_h) - f''(x)\Big|\Big] &\le \mathbb{E}\Big[\frac{B_h^2}{2h}\sup_{\eta\in]0,1[}\Big|f''(x+\eta B_h) - f''(x)\Big|\Big] \\
&= \mathbb{E}\Big[\frac{B_1^2}{2}\sup_{\eta\in]0,1[}\Big|f''(x+\sqrt{h}\,\eta B_1) - f''(x)\Big|\Big].
\end{aligned}$$

Since $x \mapsto f''(x)$ is bounded, there exists a constant C such that

$$\frac{B_1^2}{2}\sup_{\eta\in]0,1[}\Big|f''(x+\sqrt{h}\,\eta B_1) - f''(x)\Big| \le C\,B_1^2,$$

with $\mathbb{E}[C\,B_1^2] = C < \infty$. Moreover, since $x \mapsto f''(x)$ is continuous,

$$\lim_{h\to 0}\frac{B_1^2}{2}\sup_{\eta\in]0,1[}\Big|f''(x+\sqrt{h}\,\eta B_1) - f''(x)\Big| = 0 \quad \text{a.s.}$$

By the Dominated Convergence Theorem, we obtain the desired result.

Exercise 4.

Note that by the scaling invariance, for any $h > 1$ and $x > 0$, we have

$$\mathbb{P}[\sup_{t\in\mathbb{R}^+} B_t > x] = \mathbb{P}[\sup_{t\in\mathbb{R}^+} B_{ht} > \sqrt{h}x] = \mathbb{P}[\sup_{ht\in\mathbb{R}^+} B_{ht} > \sqrt{h}x] = \mathbb{P}[\sup_{t\in\mathbb{R}^+} B_t > \sqrt{h}x].$$

Thus for any $0 < x < y < \infty$, we have $\mathbb{P}[x < \sup_{t\in\mathbb{R}^+} B_t < y] = 0$, which implies that $\mathbb{P}[\sup_{t\in\mathbb{R}^+} B_t = \infty] = \mathbb{P}[\sup_{t\in\mathbb{R}^+} B_t > 0]$.

On the other hand, we have seen in class that $\mathbb{P}[\sup_{t\in[0,1]} B_t > 0] = 2\mathbb{P}[B_1 > 0] = 1$. Since $\sup_{t\in\mathbb{R}^+} B_t > \sup_{t\in(0,1]} B_t$ almost surely, we conclude that $\mathbb{P}[\sup_{t\in\mathbb{R}^+} B_t = \infty] = \mathbb{P}[\sup_{t\in\mathbb{R}^+} B_t > 0] = 1$. By symmetry we also have $\mathbb{P}[\inf_{t\in\mathbb{R}^+} B_t = -\infty] = 1$

Exercise 5.

(a) First, we compute

$$\mathbb{E}\left[\sum_n \frac{|Y_n|}{n^2}\right] = \sum_n \frac{1}{n^2}\mathbb{E}[|Y_n|] \leqslant \sum_n \frac{1}{n^2} < \infty,$$

where we used the fact that $\mathbb{E}[|Y_n|] \leqslant 1$.
Thus, the random variable $\sum_n \frac{|Y_n|}{n^2}$ is almost surely finite, and so, for $N > M \in \mathbb{N}$, we have

$$\|X^N - X^M\|_\infty \leqslant \sum_{n=M+1}^{N} \frac{1}{n^2}|Y_n|\|\sin(n\pi\cdot)\|_\infty = \sum_{n=M+1}^{N} \frac{|Y_n|}{n^2} \overset{N,M\mapsto\infty}{\longrightarrow} 0.$$

Therefore, $(X^N)_N$ is almost surely a Cauchy sequence in $\mathcal{C}[0,1]$, and we conclude that it converges to a limit denoted by X.

(b) Recall that the limit of a uniformly convergent sequence of continuous functions is again a continuous function. Therefore, the result proved above shows that $(X_t)_{t\in[0,1]}$ is almost surely continuous. The expectation of $X(t)$ is, of course, zero (by Fubini's theorem), and its variance satisfies

$$\mathbb{E}[X(t)^2] = \sum_{n,m} \frac{1}{n^2m^2}\sin(n\pi t)\sin(m\pi t)\mathbb{E}[Y_nY_m] = \sum_n \frac{1}{n^4}\sin^2(n\pi t).$$

The same argument as above gives

$$\mathbb{E}[X(t)X(s)] = \sum_{n,m} \frac{1}{n^2m^2}\sin(n\pi t)\sin(m\pi s)\mathbb{E}[Y_nY_m] = \sum_n \frac{1}{n^4}\sin(n\pi t)\sin(n\pi s).$$

Thus we have the covariance structure $(C(s,t))_{s,t\in[0,1]} = \left(\sum_n \frac{1}{n^4}\sin(n\pi t)\sin(n\pi s)\right)_{s,t}$.