Problem Set 9 April 15, 2025

Exercise 1.

Let $(B_t,\, t \geqslant 0)$ be a standard Brownian motion and let $t > 0$. Define

$$T_t = \inf\{s > t : B_s = 0\} \quad \text{and} \quad L_t = \sup\{s \leqslant t : B_s = 0\}.$$

(a) For $t_1 > t$, show that $\mathbb{P}\{T_t \leqslant t_1\} = \frac{2}{\pi} \arccos \sqrt{\frac{t}{t_1}}$.

(b) For $t_0 < t < t_1$, show that $\mathbb{P}\{L_t < t_0,\, T_t > t_1\} = \frac{2}{\pi} \arcsin \sqrt{\frac{t_0}{t_1}}$.

Exercise 2.

Let $(B_t,\, t \geqslant 0)$ be a standard Brownian motion. Define $U_t = e^{-t} B_{e^{2t}}$ for $t > 0$, and $V_t = B_t - tB_1$ for $0 < t < 1$. Calculate

$$\mathbb{E}[U_t U_s] \quad \text{and} \quad \mathbb{E}[V_t V_s].$$

Compare the distribution of V_t with the conditional distribution from Exercise 3, Problem Set 8.

Exercise 3.

Let $(B_t,\, t \geqslant 0)$ be a standard Brownian motion. Define $W_t = \int_0^t B_s \, ds$. Determine the mean and variance of W_t.

Supplementary Exercise

Exercise 4. Extinction speed of branching processes

Let (Z_n) be a branching process with the reproduction law μ. Assume that $\sum_{k \geqslant 1} k^2 \mu[k] < \infty$, $\mu[1] < 1$ and let $m = \sum_{k \geqslant 1} k\mu[k] < \infty$. Prove that if $m < 1$, then the population dies out exponentially fast. More precisely, show that there exist $\infty > c_\star, C > 0$ such that

$$\mathbb{P}[Z_n > 0] \overset{n \to \infty}{\sim} c_\star \, m^n.$$

For this, proceed as follows:

- Recall that the probability generating function of Z_n, denoted g_n, is equal to $g^{\circ n}$, the n-fold composition of the generating function g of the law μ. Using this fact, convince yourself that

$$\mathbb{P}[Z_{n+1} > 0] = 1 - g(1 - (1 - g_n(0))).$$

- Usa a Taylor expansion of g around 1 in the previous expression to prove that there exists $C \in (0, \infty)$ such that

$$m(1 - g_n(0)) - C(1 - g_n(0))^2 \leqslant 1 - g_{n+1}(0) \leqslant m(1 - g_n(0)).$$

- Using this show that there exists $c_\star \in (0, 1]$ such that $m^{-n}\mathbb{P}[Z_n > 0]$ converges to $c_\star$ as n tends to infinity.