Problem Set 8 April 8, 2025

Exercise 1.

Let (X_n) be a branching process satisfying the assumptions A and B from the course, and let Z be the random variable describing the reproduction law of the individuals. Assume also that $X_0 = 1$. Define

$$T := \inf\{n > 0 : X_n = 0\} \quad \text{and} \quad Y := \sum_{n\in\mathbb{N}} X_n.$$

(a) Show that $\mathbb{P}\{Y < +\infty\} = \mathbb{P}\{T < +\infty\}$.

(b) Let g_Y be the generating function of Y. Show that for every $s \in [0,1]$, the number $g_Y(s)$ is the unique solution x of the equation

$$x = s\, g_Z(x), \qquad 0 \leq x \leq \alpha,$$

where g_Z is the generating function of Z and α is the extinction probability.

Exercise 2. Markov Property

Let (B_t) be a standard Brownian motion. Given $0 \leq t_1 < \cdots < t_n < t$, show that

$$\mathbb{P}\{B_t \leq x \mid B_{t_n} = x_n, \ldots, B_{t_1} = x_1\} = \mathbb{P}\{B_t \leq x \mid B_{t_n} = x_n\},$$

for all x, $x_1, \ldots, x_n \in \mathbb{R}$.

Exercise 3.

Let (B_t) be a standard Brownian motion. For $t_1 < t < t_2$, show that the conditional distribution of B_t given $B_{t_1} = a$ and $B_{t_2} = b$ is Gaussian, with mean and variance

$$a + (b-a)\frac{t - t_1}{t_2 - t_1} \quad \text{and} \quad \frac{(t_2 - t)(t - t_1)}{t_2 - t_1},$$

respectively.

Exercise 4.

Let (B_t) be a standard Brownian motion. Show that

$$\mathbb{E}[B_s B_t] = \min(s,t),$$

for all $s, t \geq 0$.

Exercise 5. Lévy's Downward Theorem

Let X be an integrable random variable and let $(X_n,\, n \in \mathbb{N}^*)$ be an arbitrary sequence of random variables. For $k \in \mathbb{N}^*$, set $Y_k = (X_k, X_{k+1}, X_{k+2}, \dots)$ and $Z_k = \mathbb{E}[X|Y_k]$.

(a) For $N \in \mathbb{N}^*$, show that $(Z_{N-k+1},\, k = 1,\dots,N)$ is a martingale relative to the observation process $(Y_{N-k+1},\, k = 1,\dots,N)$.

(b) By applying the Upcrossing Lemma to the martingale $(Z_{N-k+1},\, k = 1,\dots,N)$, for each fixed $N \in \mathbb{N}^*$ and $a < b$, show that the number of upcrossings of any fixed interval $[a, b]$ by the sequence $(Z_k,\, k \in \mathbb{N}^*)$ is finite.

(c) Deduce that there is a random variable Z such that $\lim_{k\to\infty} Z_k = Z$ a.s., that is

$$Z = \lim_{k\to\infty} \mathbb{E}[X|X_k, X_{k+1}, X_{k+2}, \dots] \qquad a.s.$$