Problem Set 7 April 1, 2025

Exercise 1.

Let Z be a random variable taking values in $\mathbb{N}$ such that $\mathbb{P}\{Z = j\} = p_j$, for all $j \in \mathbb{N}$, and let $g_Z(\cdot)$ be the generating function of Z. Establish the following properties:

(a) The series $\sum_{j=0}^{+\infty} p_j s^j$ converges uniformly on $[-1, 1]$;

(b) $g_Z(s) = \mathbb{E}[s^Z]$;

(c) If Z and Z' are two independent random variables taking values in $\mathbb{N}$, then $g_{Z+Z'}(s) = g_Z(s)g_{Z'}(s)$;

(d) If $\sum_{j=0}^{+\infty} j\, p_j < +\infty$, then $g'_Z(1) = \mathbb{E}[Z]$, and if $\sum_{j=0}^{+\infty} j^2\, p_j < +\infty$, then $\mathrm{Var}(Z) = g''_Z(1) + \mathbb{E}[Z] - \big(\mathbb{E}[Z]\big)^2$.

Exercise 2.

Let (X_n) be a branching process, where the reproduction law of an individual is given by the random variable Z with generating function g_Z. Let α be the extinction probability of the population. Define $S_n = \alpha^{X_n}$. Show that (S_n) is a martingale with respect to (X_n).

Exercise 3. Overworked Server

A server takes one minute to serve each client. During minute n, the number Z_n of clients who arrive and join the queue to be served is a random variable. We assume that these random variables are independent and that

$$\mathbb{P}\{Z_n = j\} = p_j, \quad j \in \mathbb{N}, \quad \text{and} \quad \mathbb{E}[Z_n] < +\infty, \quad \forall n \in \mathbb{N}.$$

The server will only be able to take a break when no client is still waiting in the line. What is the probability that the server will be able to take a break?

Numerical application: $p_0 = 0.2$, $p_1 = 0.2$, and $p_2 = 0.6$.

Supplementary Exercise

Exercise 4. Consequences of the a.s.-submartingale convergence theorem

(a) Prove that if $(X_n,\, n \geqslant 1)$ is a super-martingale such that $\sup_n \mathbb{E}[X_n^-] < \infty$, then (X_n) converges a.s.

(b) Deduce from (a) that $(X_n,\, n \geqslant 1)$ converges almost surely under any one of the following assumptions:

1. (X_n) is a non-negative super-martingale;
2. (X_n) is a sub-martingale satisfying $\sup_n \mathbb{E}[X_n^+] < \infty$;
3. (X_n) is a non-positive sub-martingale.

Exercise 5. **Some counterexamples**

Give an example of a sequence of random variables $(X_n,\, n \geqslant 1)$ such that:

1. (X_n) is a martingale that converges a.s. convergence but not in L^1.
2. $(X_n)_n$ satisfies $\sup_n \mathbb{E}[|X_n|] < \infty$ but not $\limsup_n X_n < \infty$ a.s.