Series 6 March 25, 2025

Exercise 1.

Let $(X_n,\, n \in \mathbb{N}^*)$ be a sequence of independent random variables whose distributions are defined by

$$\mathbb{P}\{X_n = 1\} = \frac{1}{n} \quad \text{and} \quad \mathbb{P}\{X_n = 0\} = 1 - \frac{1}{n}.$$

Show that X_n converges to 0 in probability, but that X_n does not converge almost surely.

Exercise 2.

Let (S_n) be a martingale (with respect to (X_n)) such that

$$\mathbb{E}[|S_n|^p] < +\infty,$$

for every $n \in \mathbb{N}$, where $p > 1$. Show that

$$\mathbb{E}\left[\max_{0 \leqslant k \leqslant n} |S_k|^p\right] \leqslant \left(\frac{p}{p-1}\right)^p \mathbb{E}[|S_n|^p].$$

Supplementary Exercise

Exercise 3. Summability of sequences with random signs

Let $(a_n)_n$ be a sequence of real numbers. Let $(X_n,\, n \geqslant 1)$ be i.i.d. Bernoulli random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, that is, $\mathbb{P}\{X_n = 1\} = \mathbb{P}\{X_n = -1\} = 1/2$. Prove that

$$\sum_{n=1}^{\infty} a_n X_n \text{ converges in } L^2(\Omega) \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} a_n^2 < \infty.$$