Problem Set 5 March 18, 2025

Exercise 1. (Cauchy Criterion for Almost Sure Convergence)

Let (S_n) be a sequence of random variables such that for every $\varepsilon > 0$,

$$\lim_{n\to+\infty} \mathbb{P}\left\{\sup_{i\geqslant n}|S_i - S_n| > \varepsilon\right\} = 0.$$

Show that there exists a random variable S such that

$$\lim_{n\to+\infty} S_n = S \quad \text{almost surely.}$$

Exercise 2.

Let (X_n) be a sequence of random variables taking values in $\mathbb{N}$. We say that (X_n) is a *Markov chain* if there exists a matrix $P = (p_{i,j})$, called the *transition matrix*, such that for every $n \in \mathbb{N}$,

$$\mathbb{P}\{X_{n+1} = j_{n+1} \mid X_n = j_n, \dots, X_1 = j_1\} = \mathbb{P}\{X_{n+1} = j_{n+1} \mid X_n = j_n\} = p_{j_n,j_{n+1}}.$$

Let $f : \mathbb{N} \to \mathbb{R}$ be a bounded function such that for every i,

$$\sum_j p_{i,j} f(j) = f(i).$$

(a) Show that $(Y_n = f(X_n),\, n \in \mathbb{N})$ is a martingale with respect to (X_n).

(b) Suppose that for every i and j,

$$\mathbb{P}\{X_n = j \text{ for infinitely many } n \mid X_1 = i\} = 1.$$

Then show that f is constant.

Supplementary Exercise

Exercise 3. Wald's identity

Let $(X_n)_{n\in\mathbb{N}}$ be i.i.d. integrable random variables and let τ be a stopping time relative to (X_n) such that $\mathbb{E}[\tau] < \infty$. Explain why S_τ is well-defined and prove that

$$\mathbb{E}[S_\tau] = \mathbb{E}[X_1]\,\mathbb{E}[\tau].$$