Problem Set 4 March 11, 2025

Exercise 1.

Let $(X_i,\ i \geqslant 0)$ be a sequence of random variables such that for all $i \in \mathbb{N}$, $\mathbb{E}[|X_i|] < +\infty$ and $\mathbb{E}[X_{i+1}|X_0, \dots, X_i] = 0$. Define $S_0 = X_0$ and for all $n \geqslant 0$,

$$S_{n+1} = X_0 + \sum_{i=0}^{n} X_{i+1} f_i(X_0, \dots, X_i),$$

where the functions $f_i : \mathbb{R}^{i+1} \to \mathbb{R}$ are continuous and bounded. Show that $(S_n,\ n \geqslant 0)$ is a martingale with respect to $(X_n,\ n \geqslant 0)$.

Exercise 2.(Doob's Decomposition)

Let $(S_n, n \geqslant 1)$ be a submartingale with respect to (X_n). Show that there exists a unique decomposition

$$S_n = M_n + A_n$$

where (M_n) is a martingale with respect to (X_n) and (A_n) is an increasing sequence of random variables such that $A_1 = 0$ and A_{n+1} is a function of $(X_1, \dots, X_n)$ for all $n \geqslant 1$.

Exercise 3.

We revisit the situation from Exercise 2 of Problem Set 2. Let T be the number of balls drawn until the first green ball appears. Show that

$$\mathbb{E}\left[\frac{1}{T+2}\right] = \frac{1}{4}.$$

Supplementary Exercises

Exercise 4.(Integrability criterion for a stopping time)

Let T be a stopping time relative to (X_n). Suppose there exists $\varepsilon > 0$ and $N \geqslant 1$ such that for all $n \geqslant 0$,

$$\mathbb{P}[T \leqslant n + N | X_1, \dots, X_n] > \varepsilon \quad \text{a.s.}$$

Prove that for all $k \geqslant 0$,

$$\mathbb{P}[T > kN] \leqslant (1 - \varepsilon)^k.$$

Deduce that $T < \infty$ almost surely and that for all $p \geqslant 1$, $\mathbb{E}[T^p] < \infty$, and also that $\mathbb{E}[e^{\lambda T}] < \infty$ if $\lambda > 0$ is sufficiently small.

Exercise 5.(Reverse Optional stopping theorem)

Let (X_n) be a sequence of random variables such that for all $n \geqslant 0$, $\mathbb{E}[|X_n|] < \infty$. Show that if $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$ for all bounded stopping times τ relative to (X_n), then $(X_n)_n$ is a martingale.