Problem Set 3 March 4, 2025

Exercise 1.

Let X be a random variable and Y and Z be random vectors such that (X, Y) is independent of Z. Show that

$$\mathbb{E}[X|Y, Z] = \mathbb{E}[X|Y].$$

Exercise 2.

(a) Let (S_n) be a martingale relative to (X_n). Show that for all integers $k \leqslant l \leqslant m$,

$$\mathbb{E}[(S_m - S_l)S_k] = 0.$$

(b) Let $(X_n,\, n \geqslant 1)$ be a sequence of random variables and define $S_n = X_1 + \cdots + X_n$. Show that if (S_n) is a martingale relative to (X_n), then $\mathbb{E}[X_i X_j] = 0$ for all $i \neq j$.

Exercise 3.

(a) Let $Y_1, \ldots, Y_n$ be a sequence of random variables. To each stopping time T relative to (Y_n) with values in $\{1, \ldots, n\}$, there corresponds a sequence of sets $B_1, \ldots, B_n$ such that

$$\{T = k\} = \{(Y_1, \ldots, Y_k) \in B_k\}.$$

Conversely, given a sequence of sets $B_1, \ldots, B_n$, under what condition does this sequence correspond to a stopping time?

(b) Let $(X_n,\, n \geqslant 1)$ be a sequence of random variables. Define $S_n = X_1 + \cdots + X_n$ and

$$T = \inf\{n \geqslant 1 : S_n \in [1, 2)\}.$$

Show from the definition that T is a stopping time relative to (X_n).

Supplementary Exercises

Exercise 4.

Let τ_1, τ_2 be two stopping times relative to $(Y_n)_n$. Show that $\tau_1 + \tau_2$ is a stopping time.

Exercise 5.

Suppose we roll two dice repetitively until the sum of the two numbers is even. What is the law of the sum of the two numbers on the first roll? On the last roll?

In order to answer these questions, construct an appropriate probability space. If you define a stopping time, check that it is indeed one and specify relative to which sequence of random variables.