Problem Set 2 February 25, 2025

Exercise 1.

Let $(X_n,\, n \geqslant 1)$ be an i.i.d. sequence of random variables (discrete or continuous) such that $\mathbb{E}(|X_1|) < +\infty$, and let $N \in \mathbb{N}$ be fixed. Define $Z_n = X_1 + \cdots + X_n$, $Y_n = \frac{1}{n} Z_n$, $R_n = Z_{N-n}$, and $S_n = Y_{N-n}$ for $n \in \{0, \ldots, N-1\}$. Show that (S_n) is a martingale relative to (R_n).

Exercise 2.

A box contains red and green balls. At each step, a ball is drawn at random, then returned to the box along with another ball of the same color. Initially, the box contains one red ball and one green ball. Let R_n be the number of red balls in the box after n steps. Show that

$$\left(S_n = \frac{R_n}{n+2} \right)$$

is a martingale relative to (R_n).

Exercise 3.

Let Z be a random variable uniformly distributed over $[0,1]$. For $n \geqslant 1$ and $k \in \{0, \ldots, 2^n\}$, define $X_n = k2^{-n}$ if $k2^{-n} \leqslant Z < (k+1)2^{-n}$. Let $f : [0,1] \to \mathbb{R}$ be a bounded function and $Y_n = 2^n(f(X_n + 2^{-n}) - f(X_n))$. Show that (Y_n) is a martingale relative to (X_n).

Supplementary Exercise

Exercise 4.

Let $(X_n,\, n \geqslant 1)$ and $(Y_n,\, n \geqslant 1)$ be a sub- and supermartingale, respectively, relative to (Z_n). Prove the following statements.

1. If the sequence $(\mathbb{E}[X_n])$ is constant, then (X_n) is a martingale. Establish the same conclusion for (Y_n).
2. For any $a \in \mathbb{R}$, show that:
   a) $(\max(a, X_n))$ is a submartingale (relative to (Z_n));
   b) $(\min(a, Y_n))$ is a supermartingale (relative to (Z_n)).

(c) Assume that all the X_n have the same law. Show that (X_n) is a martingale (which is called **equidistributed**). [Note. It is possible to show that in fact, $X_1 = X_2 = \cdots$]