Problem Set 1: Expectation and Conditional Probability February 18, 2025

Exercise 1.

Let (X, Y, Z) be a discrete (or continuous) random vector defined on a probability space, and let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function. Show the following:

(a) $\mathbb{E}[\alpha Y + \beta Z | X] = \alpha\, \mathbb{E}[Y|X] + \beta\, \mathbb{E}[Z|X]$, for all α, $\beta \in \mathbb{R}$;

(b) if $Y \leqslant Z$, then $\mathbb{E}[Y|X] \leqslant \mathbb{E}[Z|X]$;

(c) $\mathbb{E}[\mathbb{E}[Z|X, Y]|X] = \mathbb{E}[Z|X]$;

(d) $\mathbb{E}[Y f(X)|X] = \mathbb{E}[Y|X] f(X)$;

(e) if X and Y are independent, then $\mathbb{E}[Y|X] = \mathbb{E}[Y]$;

(f) if $Y = f(X)$, then $\mathbb{E}[Y|X] = Y$;

(g) if $g : \mathbb{R} \to \mathbb{R}$ is a convex function and $\mathbb{E}[|g(Y)|] < \infty$, then $g(\mathbb{E}[Y|X]) \leqslant \mathbb{E}[g(Y)|X]$.

Exercise 2.

Let $(X_k,\ k \geqslant 1)$ be a sequence of independent random variables uniformly distributed over $[0, 1]$. Let $x \in]0, 1[$ and $N = \min\{n \geqslant 1 : X_1 + \cdots + X_n > x\}$. Show that

$$\mathbb{P}\{N > n\} = \frac{x^n}{n!}.$$

Supplementary Exercises

Exercise 3.

Let X_1, X_2, X_3 be three random variables satisfying $\mathbb{E}[|X_i|] < \infty$, $i = 1, 2, 3$. Show that if X_1 is independent of (X_2, X_3) and $\mathbb{E}[|X_1 X_2|] < \infty$, then

$$\mathbb{E}[X_1 X_2 | X_3] = \mathbb{E}[X_1]\mathbb{E}[X_2|X_3] \ \ a.s..$$

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Exercise 4.

Let X, Y be two random variables, whose joint law, that is, the law of (X, Y), has a density $h(x, y)$, and assume that $\mathbb{E}[|X|] < \infty$. Let $I \in \mathbb{R}$ be an interval and let $y_0 \in \mathbb{R}$. For $\varepsilon > 0$, compute $\mathbb{P}\{X \in A \mid Y \in]y_0 - \varepsilon, y_0 + \varepsilon[\}$ and propose an expression for $\mathbb{E}[X|Y]$. Finally, show that

$$\mathbb{E}[X|Y] = \frac{\int_{\mathbb{R}} x\, h(x, Y)\, dx}{\int_{\mathbb{R}} h(x, Y)\, dx} \ . \tag{1}$$

Exercise 5.

Let $X_1, \dots, X_n$ be i.i.d. integrable random variables, that is, $\mathbb{E}[|X_i|] < \infty$. Show that:

(a) $\mathbb{E}[\sum_{i=1}^n X_i | X_1] = X_1 + (n-1)\mathbb{E}[X_1]$ almost surely;

(b) $\mathbb{E}[X_1 | \sum_{i=1}^n X_i] = \dfrac{1}{n} \sum_{i=1}^n X_i$ almost surely.