Problem Set 13 May 20, 2025

Exercise 1.

Assume that we know how to construct a process $(B_t^{(1)},\, t \in [0,1])$ that satisfies conditions (a) and (b) of the definition of standard Brownian motion and such that the mapping $t \mapsto B_t^{(1)}(\omega)$ is continuous on $[0,1]$. Construct a process $(B_t,\, t \geqslant 0)$ that is a standard Brownian motion such that the mapping $t \mapsto B_t(\omega)$ is a continuous function on $\mathbb{R}_+$.

Exercise 2.

The goal of this exercise is to show that the sample paths of Brownian motion are nowhere differentiable.

(a) Let $F = \{\exists\, t_0 \in [0,1] : B'_{t_0} \text{ exists and is finite}\}$, and for $N \in \mathbb{N}$ and $n \in \mathbb{N}^*$, let

$$G_{N,n} = \left\{\exists\, t_0 \in [0,1] : |B_t - B_{t_0}| \leq N|t - t_0| \text{ if } |t - t_0| \leq \frac{3}{n}\right\}.$$

Show that $F \subset \bigcup_{N\in\mathbb{N}} \bigcup_{n\in\mathbb{N}^*} G_{N,n}$.

(b) For $1 \leq k \leq n-2$, define

$$Y_{k,n} = \max_{j=0,1,2} \left| B\left(\frac{k+j}{n}\right) - B\left(\frac{k+j-1}{n}\right) \right|,$$

and

$$H_{N,n} = \bigcup_{k=1}^{n-2} \left\{ Y_{k,n} \leq \frac{5N}{n} \right\}.$$

Show that $G_{N,n} \subset H_{N,n}$.

(c) Check that $\mathbb{P}\left\{Y_{k,n} \leq \frac{5N}{n}\right\} = \left(\mathbb{P}\left\{\left|B\left(\frac{1}{n}\right)\right| \leq \frac{5N}{n}\right\}\right)^3$ and deduce that $\lim_{n\to\infty} \mathbb{P}(H_{N,n}) = 0$, and hence that $\mathbb{P}(F) = 0$.

Supplementary Exercise

Exercise 3. Zero-set of Brownian motion

Let $\mathcal{Z}$ be the zero set of a one-dimensional Brownian motion $B = (B_t,\, t \in \mathbb{R}_+)$, that is, $\mathcal{Z} = \{t \in \mathbb{R}_+ : B_t = 0\}$. Show that:

(a) $\mathcal{Z}$ is almost surely a perfect set, that is, a closed set with no isolated point;

(b) the length of $\mathcal{Z}$ is almost surely zero.