Problem Set 12 May 13, 2025

Exercise 1.

Let $(B_t,\, t \geqslant 0)$ be a standard Brownian motion.

(a) Given $a > 0$ and $b < a$, show that

$$\mathbb{P}\left\{\sup_{t\geqslant 0} \frac{b + B_t}{1+t} \geqslant a\right\} = e^{-2a(a-b)}.$$

(b) Show that the left-hand side (and hence the right-hand side as well) is equal to

$$\mathbb{P}_0\left\{\sup_{0\leqslant u\leqslant 1} B_u \geqslant a \mid B_1 = b\right\}.$$

Exercise 2.

Let $(B_t,\, t \geqslant 0)$ be a standard Brownian motion. Show that

$$\mathbb{P}_0\left\{\bigcap_{n\in\mathbb{N}} \{\exists\, t \geqslant n : B_t = 0\}\right\} = 1.$$

Exercise 3.

Let $(B_t,\, t \geqslant 0)$ be a standard Brownian motion. For $k \in \mathbb{N}^*$, define

$$C_k = \frac{2}{\pi}\int_0^\pi \left(B_t - \frac{t}{\pi} B_\pi\right) \sin(kt)\, dt.$$

(a) Compute $\mathbb{E}(C_k)$ and $\mathbb{E}(C_k C_\ell)$ for $k,\, \ell \in \mathbb{N}$.

(b) Determine the distribution of C_k for $k \in \mathbb{N}$.

(c) Are the random variables $(C_k,\, k \in \mathbb{N})$ independent? Are they independent of B_π?

(d) Let $(Z_k,\, k \in \mathbb{N})$ be i.i.d. $\mathrm{N}(0,1)$ random variables. Define

$$X_t = \frac{t}{\sqrt{\pi}}\, Z_0 + \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} \frac{\sin kt}{k}\, Z_k.$$

Show that (X_t) and (B_t) have the same law. (In particular, (X_t) is a Brownian motion. This formula is the *Paley-Wiener representation* of Brownian motion.)