Problem Set 11 May 6, 2025

Exercise 1.

Let $(B_t,\, t \geqslant 0)$ be a standard Brownian motion, $\mu \in \mathbb{R}$, and define $Z_t = B_t + \mu t$. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function with bounded second derivative. Show that

$$\lim_{h \downarrow 0} \frac{\mathbb{E}[f(Z_{t+h}) \mid Z_t = x] - f(x)}{h} = \mu f'(x) + \frac{f''(x)}{2}.$$

Exercise 2.

Let $(B_t,\, t \geqslant 0)$ be a standard Brownian motion. Define $M_t = \max_{0 \leqslant u \leqslant t} B_u$.

(a) Find the joint density of $(M_t,\, M_t - B_t)$.

(b) Find the density of $M_t - B_t$.

(c) Show that $\mathbb{P}\{M_t > \xi \mid B_t = M_t\} = \exp\left(-\frac{\xi^2}{2t}\right)$.

Supplementary Exercises

Exercise 3.

Let $B = (B_t,\, t \in [0,1])$ be a standard Brownian motion. Use B to construct a stochastic process $W = (W_t,\, t \in [0,1])$ such that for all $n \in \mathbb{N}$, $0 < t_1 < \cdots < t_n$ and subsets $A_1, \ldots, A_n$ of $\mathbb{R}$,

$$P\{B_{t_1} \in A_1, \ldots, B_{t_n} \in A_n\} = P\{W_{t_1} \in A_1, \ldots, W_{t_n} \in A_n\},$$

(that is, B and W have the same marginal distributions), but W is almost surely discontinuous.

Exercise 4. Brownian motion has no interval of monotonicity

Let $B = (B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion. Show that almost surely, the samle paths of B have no interval of monotonicity, that is, almost surely, for all $0 < a < b < \infty$, $t \mapsto B_t$ is not monotone on $[a, b]$.

Exercise 5. Passage times at fixed levels for a Brownian motion

Let $a, b > 0$ and set

$$T_{-a,b} = \inf\{t \geqslant 0 : B_t \in \{-a, b\}\}, \qquad T_b = \inf\{t \geqslant 0 : B_t = b\}.$$

Show that $P\{T_{-a,b} < \infty\} = P\{T_b < \infty\} = 1$.