Solution Sheet n°9

Solution of exercise 1:

(1. $\Longrightarrow$ 2.) notice that:

$$\forall \vec{y} \in \mathbf{M}\big(\exists x\ \psi(x,\vec{y}) \leftarrow \exists x \in \mathbf{M}\ \psi(x,\vec{y})\big)$$

is obvious, we show:

$$\forall \vec{y} \in \mathbf{M}\big(\exists x\ \psi(x,\vec{y}) \rightarrow \exists x \in \mathbf{M}\ \psi(x,\vec{y})\big).$$

Let $\varphi \in F$ be of the form $\exists x\ \psi(x,\vec{y})$. We fix $\vec{y} \in \mathbf{M}$, and assume that $\exists x\ \psi(x,\vec{y})$ holds. Since φ is absolute for $\mathbf{M}$, we have $\big(\exists x\ \psi(x,\vec{y})\big)^{\mathbf{M}}$ i.e. $\exists x \in \mathbf{M}\, \psi(x,\vec{y})^{\mathbf{M}}$ which gives $\exists x \in \mathbf{M}\, \psi(x,\vec{y})$ by absoluteness of $\psi(x,\vec{y})$.

(2. $\Longrightarrow$ 1.) the proof goes by induction on the height of $\varphi \in F$.

- If φ is atomic, then the result is obvious since $\varphi^{\mathbf{M}} := \varphi$
- If φ is of the form $\neg\psi$ or $\phi_0 \wedge \phi_1$, then the result the result follows immediately by induction since $(\neg\psi)^{\mathbf{M}} := \neg(\psi^{\mathbf{M}})$ and $(\phi_0 \wedge \phi_1)^{\mathbf{M}} := \phi_0^{\mathbf{M}} \wedge \phi_1^{\mathbf{M}}$
- If φ is of the form $\exists x\ \psi(x,\vec{y})$, then fix $\vec{y} \in \mathbf{M}$ and notice that

$$\begin{aligned}\big(\exists x\ \psi(x,\vec{y})\big)^{\mathbf{M}} &\longleftrightarrow \exists x \in \mathbf{M}\ \psi(x,\vec{y})^{\mathbf{M}}\\ &\longleftrightarrow^{1} \exists x \in \mathbf{M}\ \psi(x,\vec{y})\\ &\longleftrightarrow^{2} \exists x\ \psi(x,\vec{y}).\end{aligned}$$

□

Solution of exercise 2:

1. W.l.o.g. we may assume that F is closed under sub-formulae. We fix α and look for β.

   Let $\widetilde{F}$ be the set of formulae in F which are of the form $\exists x \psi(x,\vec{y})$, with $\vec{y} = (y_1,\ldots,y_n)$. For each $\varphi \in \widetilde{F}$ we define the functional $G_\varphi : V^n \to \mathbf{ON}$ by:

$$G_\varphi(y_1,\ldots,y_n) = \begin{cases}0 & \textit{if } \neg\varphi(y_1,\ldots,y_n)\\ \min\{\eta \in \mathbf{ON} \mid \exists x \in V_\eta\ \psi(x,y_1,\ldots,y_n)\} & \textit{if } \varphi(y_1,\ldots,y_n).\end{cases}$$

   We then define the functional $H_\varphi : \mathbf{ON} \to \mathbf{ON}$ by:

$$H_\varphi(\xi) = \sup\{G_\varphi(y_1,\ldots,y_n) \mid y_1,\ldots,y_n \in V_\xi\}$$

[1] $\psi(x,\vec{y})$ is absolute for $\mathbf{M}$, by induction hypothesis.
[2] by hypothesis.

We define inductively a strictly increasing sequence of ordinals $(\beta_n)_{n\in\omega}$ by:

$$\begin{array}{lcl} \beta_0 & = & \alpha \\ \beta_{n+1} & = & \sup\left(\{\beta_n+1\}\cup\{H_\varphi(\beta_n)\mid \varphi\in\widetilde{F}\}\right) \end{array}$$

and take $\beta=\sup(\beta_n)_{n\in\omega}$. By construction we have

$$\alpha=\beta_0<\beta_1<\ldots<\beta_n<\beta_{n+1}<\ldots<\beta$$

hence β is a limit ordinal.

Suppose that φ is of the form $\exists x\ \psi(x,\vec{y})$ and consider any $\vec{y}=(y_1,\ldots,y_n)$ such that $y_1\in V_\beta,\ldots,y_n\in V_\beta\ \exists x\ \psi(x,\vec{y})$ holds (in V). Then since β is a limit ordinal, there exists some ordinal $\beta_p<\beta$ such that $y_1\in V_{\beta_p},\ldots,y_n\in V_{\beta_p}$.

Now, by definition of H_φ, there exists $x\in V_{H_\varphi(\beta_p)}$ such that $\psi(x,\vec{y})$ holds (in V). Therefore there exists $x\in V_\beta$ such that $\psi(x,\vec{y})$ holds (in V). By Exercise 1, it follows that all formulae in F are absolute for V_β.

2. Since M_0 is a set, there exists some ordinal α such that $m_0\subseteq V_\alpha$. By point 1, there exists some $\beta>\alpha$ such that all formulae in F are absolute for V_β. Let $M=V_\beta$.

□

Solution of exercise 3:

Proof of the theorem: We first assume that the list of formulae contains the Axiom of Extensionality and is closed under sub-formulae.

For some α, the set M_0 belongs V_α. Hence, by previous Exercise, there exists $\beta>\alpha$ such that $\varphi_1,\ldots,\varphi_n$ are absolute for V_β. Using the axiom of choice, fix a well-ordering $\triangleleft$ on V_β, and for each integer $1\leq i\leq n$ and formula φ_i with m_i free variables we define $H_i\ :\ V_\beta^{m_i}\longrightarrow V_\beta$:

- if φ_i is not an existential formula, or if φ_i is of the form $\exists x\ \varphi_j(x,y_1,\ldots,y_{m_i})$ but $V_\beta\not\models\exists x\ \varphi_j(x,y_1,\ldots,y_{m_i})$:

  $H_i(y_1,\ldots,y_{m_i}):=$ *the $\triangleleft$-least element of V_β*

- if φ_i is of the form $\exists x\ \varphi_j(x,y_1,\ldots,y_{m_i})$ and $V_\beta\models\exists x\ \varphi_j(x,y_1,\ldots,y_{m_i})$:

  $H_i(y_1,\ldots,y_{m_i}):=$ *the $\triangleleft$-least $x\in V_\beta$ such that $\varphi_j(x,y_1,\ldots,y_{m_i})$ holds.*

We then inductively define:

- $\overline{M_0}=M_0$,
- $\overline{M_{m+1}}=\overline{M_m}\cup\displaystyle\bigcup_{1\leq i\leq n}\{H_i(y_1,\ldots,y_{m_i})\mid y_1\in\overline{M_m},\ldots,y_{m_i}\in\overline{M_m}\}$,
- $\overline{M}=\displaystyle\bigcup_{m<\omega}\overline{M_m}$.

By construction $\overline{M}$ is closed under all functions H_i (any $1 \leq i \leq n$) and $|\overline{M}| \leq \max(\omega,|M_0|)$.

If φ_i is of the form $\exists x\ \varphi_j(x,y_1,\ldots,y_{m_i})$ then:

$$\forall y_1 \in \overline{M} \ldots \forall y_{m_i} \in \overline{M}\ \Big[\exists x\, \phi_j(x,y_1,\ldots,y_{m_i}) \ \leftrightarrow\ \exists x \in V_\beta\ \phi_j(x,y_1,\ldots,y_{m_i}) \ \leftrightarrow\ \exists x \in \overline{M}\ \phi_j(x,y_1,\ldots,y_{m_i})\Big]$$

By Exercise 1, it follows that every formula φ_i (any $1 \leq i \leq n$) is absolute for $\overline{M}$. In particular, since we assumed that the Axiom of Extensionality was among this list of formulae, it follows that *Extensionality*$^{\overline{M}}$, hence $\overline{M}$ is extensional so we can make use of the Mostowski collapse Theorem to obtain M as the Mostowski collapse of $\overline{M}$, i.e. M is transitive and there exists some isomorphism $g : \overline{M} \longleftrightarrow M$. Therefore, $|M| = |\overline{M}|$ and for every formula φ_i (any $1 \leq i \leq n$) we have for every $y_1 \in \overline{M},\ldots,y_{m_i} \in \overline{M}$:

$$\varphi_i(y_1,\ldots,y_{m_i})^{\overline{M}} \ \longleftrightarrow\ \varphi_i(g(y_1),\ldots,g(y_{m_i}))^M.$$

Therefore φ_i is absolute for $\overline{M}$ implies φ_i is absolute for M (any $1 \leq i \leq n$). □

Proof of the corollary: immediate, by taking M_0 to be any finite of countable transitive set (e.g. $M_0 = \emptyset$).

□

Solution of exercise 4: Since the lecture on Set Theory started, we only proved finitely many results and each time we only made use of finitely many axioms from ZFC – even though we used both axiom schemas, we only needed finitely many instances of each of them. We should remain cautious however, and emphasize that this remark only concerns lemmas and theorems, but not theorem schemas – such as the transfinite recursion theorem for instance. Nevertheless, we can make the conjunction of all the finitely many axioms that all our proofs of lemmas and theorems required. This way we obtain a single formula ϕ. By the Corollary of Exercise 3. we get a transitive countable set M that satisfies every single lemma and theorem that we proved. Therefore, in M, there are a cardinal ω_1 and a set $\mathcal{P}(\omega)$ that satisfy everything we proved about them. In particular they both satisfy that they are uncountable although they are both countable sets in V, since they are elements of a transitive countable set. This means that in V there exist a bijection $f : \omega_1^M \longleftrightarrow \omega$ and a bijection $g : \mathcal{P}(\omega)^M \longleftrightarrow \omega$.

This shows that the property of "being countable" is not absolute.

□