Solution Sheet n°8

1. *For any infinite cardinal κ, $H_\kappa \subseteq V_\kappa$.*

   For any set x and for $t = cl(x)$, we show that the set of ordinals $S = \{\mathrm{rk}(y) : y \in t\}$ is an ordinal. It is enough to show that S is transitive. Now, for a set of ordinals S, S is transitive if and only if the smallest ordinal which does not belong to S is S itself (this ordinal exists since S is a set, thus it cannot be cofinal in the class of all ordinals).

   Indeed, let α be the smallest ordinal does not belong to S, then $\alpha \subseteq S$. But if $\alpha \neq S$, then there exists $\beta \in S \setminus \alpha$ and therefore $\alpha < \beta \in S$, i.e. S is not transitive. On the other hand, if S is not transitive, there exists $\beta < \gamma \in S$ with $\beta \notin S$ and therefore if α is the smallest ordinal which does not belong to S we have $\alpha \leq \beta$ and $\gamma \in S \setminus \beta \subseteq S \setminus \alpha$, therefore $\alpha \neq S$.

   Therefore, let α be the smallest ordinal which does not belong to S. Suppose towards contradiction that $\alpha \subsetneq S$. Let $\beta = \min S \setminus \alpha$ and let $y \in t$ with $\mathrm{rk}(y) = \beta$. By definition of β, $\beta > \alpha$. Since t is transitive, for all $z \in y$, we have $\mathrm{rk}(z) \in S$ and, by definition of α, $\mathrm{rk}(z) \neq \alpha$. Since $\mathrm{rk}(z) < \mathrm{rk}(y) = \beta$, by minimality of β it is impossible that $\mathrm{rk}(z) > \alpha$. Therefore for all $z \in y$ we have $\mathrm{rk}(z) < \alpha$. Now, by definition of the rank, $\mathrm{rk}(y) = \sup\{\mathrm{rk}(z) + 1 \mid z \in y\} \leq \alpha$, i.e. $\beta \leq \alpha$, a contradiction. Therefore, $\alpha = S$ and therefore S is an ordinal. Indeed, this ordinal is precisely the rank of x (exercise!).

   Now if $x \in H_\kappa$ for κ an infinite cardinal, then $|t| = |cl(x)| < \kappa$. The ordinal $\alpha = \{\mathrm{rk}(y) \mid y \in cl(x)\}$ is therefore such that $|\alpha| \leq |cl(x)| < \kappa$ since the function $\mathrm{rk} : cl(x) \to \alpha$ is surjective (recall that $\alpha = \mathrm{rk}(x)$). Therefore, $\mathrm{rk}(x) = \alpha \leq |cl(x)| < \kappa$, i.e. $x \in V_\kappa$.

2. *(AC) If κ is an infinite regular cardinal, then we have $\forall x(x \in H_\kappa \leftrightarrow x \subseteq H_\kappa \wedge |x| < \kappa)$.*

   Let $x \in H_\kappa$. For all $y \in x$, the fact that $cl(y) \subseteq cl(x)$ implies $|cl(y)| < \kappa$ and therefore $y \in H_\kappa$. Thus, $x \subseteq H_\kappa$ and furthermore, since $x \subseteq cl(x)$, we also have $|x| < \kappa$.

   On the other hand, let $x \subseteq H_\kappa$ with $|x| < \kappa$. Let $\lambda = |x|$ and f be a bijection between λ and x. Consider the function $g : \lambda \to \kappa$ defined by $g(\alpha) = |cl(f(\alpha))|$, for all $\alpha < \lambda$. The images of g are indeed in κ, since $x \subseteq H_\kappa$ and therefore for each $y \in x$, we have $|cl(y)| < \kappa$. By the regularity of κ the function g is not cofinal and there exists therefore, a cardinal $\mu < \kappa$ such that $g(\alpha) \leq \mu$ for all α, i.e. $|cl(y)| \leq \mu$ for all $y \in x$. Let $\theta = \max\{\mu, \lambda\} < \kappa$. By the axiom of choice, a union of θ sets of cardinality at most θ has cardinality at most θ (Exercise 2 of the Sheet 4). Thus,

$$|cl(x)| = \left| x \ \cup \ \bigcup\{cl(y) : y \in x\} \right| \leq \lambda \oplus \theta < \kappa.$$

3. *(ZFC) If κ is a regular uncountable cardinal regular, then H_κ is model of $ZFC - P$.*

   We begin by noticing the following properties (in ZF) of H_κ, for an infinite κ:

   (a) H_κ is transitive;

   (b) $H_\kappa \cap \mathbf{ON} = \kappa$;

   (c) if $x \in H_\kappa$, then $\bigcup x \in H_\kappa$;

   (d) if $x, y \in H_\kappa$, then $\{x, y\} \in H_\kappa$;

   (e) if $x \in H_\kappa$ and $y \subseteq x$, then $y \in H_\kappa$;

For (a), simply notice that if $x \in y$, then $cl(x) \subseteq cl(y)$. For (b), it is enough to remark that $\alpha = cl(\alpha)$ for every ordinal α, since the ordinals are transitive. Point (c) follows from the fact that $\bigcup x \subseteq cl(x)$. For (d), observe that $cl(\{x,y\}) = cl(x) \cup cl(y) \cup \{x,y\}$. Point (e) is a consequence of the fact that $y \subseteq x$ implies $cl(y) \subseteq cl(x)$.

Let $\kappa > \aleph_0$ be regular. Since H_κ is transitive, we can make use of the criteria of satisfaction for the axioms of ZFC for transitive classes (Exercise 2 of Sheet 7). Extensionality is immediate, the axiom of foundation also follows immediately (we work in $\mathbf{V} = \mathbf{WF}$). The union and pairing axioms follow by points (c) and (d) above. For replacement, use point 2. Let $f : x \to H_\kappa$ be a function of domain $x \in H_\kappa$ with image in H_κ. The cardinality of the image of f is less or equal than $|x| < \kappa$. By the previous point, the image of f belongs to H_κ. For the axiom of infinity, by point (b) we have $\omega \in \kappa = H_\kappa \cap \mathbf{ON}$. Finally, for the axiom of choice, the formula "R well orders A" is downwards absolute for the models of $ZF^- - P - Inf$. Thus, for $x \in H_\kappa$, by AC there exists $R \subseteq x \times x$ (in $\mathbf{V}$) such that "R well orders x" and therefore such that "(R well orders $x)^{H_\kappa}$". By point (d), $R \subseteq H_\kappa$ and since $|R| \leq |x \times x| = |x| < \kappa$ we have $R \in H_\kappa$ by point 2.

4. *(ZFC) If κ is a regular uncountable non strongly inaccessible cardinal, then H_κ is model of $ZFC - P + \neg P$.*

   By point (e) above, we have that for all $x \in H_\kappa$, $\mathcal{P}^{H_\kappa}(x) = \mathcal{P}(x) \cap H_\kappa = \mathcal{P}(x)$. Now, by Exercise 2 of Sheet 7 and the transitivity of H_κ, the satisfaction of the powerset axiom by H_κ is equivalent to $\forall x \in H_\kappa\ \exists c \in H_\kappa\ (\mathcal{P}(x) \cap H_\kappa \subseteq c)$ and therefore by the remark preceding point (e), it is also equivalent to $\forall x \in H_\kappa(\mathcal{P}(x) \in H_\kappa)$. But if κ is regular but not strongly inaccessible, there exists $\lambda < \kappa$ such that $2^\lambda \geq \kappa$ and by (a) $2^\lambda \notin H_\kappa$. We therefore have that "$H_\kappa \models ZFC - P + \neg P$".

   The following relative consistency results for the powerset axiom thus hold:

   Theorem. *$Con(ZFC) \to Con(ZFC - P + \neg P)$, i.e.*
   $Con(ZFC) \to$ "$ZFC - P \ \nvdash\ P$".

5. *(ZFC) If κ is a strongly inaccessible cardinal, then H_κ is a set model of ZFC.*

   By point 1, we have that $H_\kappa \subseteq V_\kappa$ for all infinite κ. If moreover κ is strongly inaccessible, i.e. regular and strongly limit, we show that $H_\kappa = V_\kappa$. Look at the proof of the lemma in the Solution of Exercise 3 of Sheet 8. We have shown that for a strongly inaccessible κ and all $\gamma < \kappa$, we have $|V_\gamma| < \kappa$. Now, if $x \in V_\alpha$ for $\alpha < \kappa$ then, by transitivity of V_α we have $cl(x) \subseteq V_\alpha$. Therefore, $|cl(x)| \leq |V_\alpha| < \kappa$ and $x \in H_\kappa$. It follows that $H_\kappa = V_\kappa$, and, by the first part of Exercise 3 of Sheet 8, that H_κ is a set model of ZFC.

   We have thus reproved the following:

   Theorem. *$ZFC + \exists\kappa(\kappa$ strongly inaccessible$)\ \vdash\ Con(ZFC)$.*

6.

   Theorem. *$Con(ZFC) \to Con(ZFC + \neg\exists\kappa(\kappa$ strongly inaccessible$))$,*
   i.e. $Con(ZFC) \to$ "$ZFC\ \nvdash\ \exists\kappa(\kappa$ strongly inaccessible$)$".

   Since for a strongly inaccessible κ, $H_\kappa = V_\kappa$, we can substitute V_κ with H_κ in the proof of this result from the Solution of Exercise 3 of Sheet 8.