Solution Sheet n°7

Solution of exercise 1:

$(a) - (p)$ See Kunen pp. 119-122 and p. 126.

Proof of the Proposition: by the previous proposition we have: for all $x \in \mathbf{M}$, "x is an ordinal in $\mathbf{M}$" iff "x is an ordinal". i.e. $\mathbf{On}^{\mathbf{M}} = \mathbf{On} \cap \mathbf{M}$. By absoluteness of $\in$ we also have $(\alpha < \beta)^{\mathbf{M}}$ iff $\alpha < \beta$. We let $\kappa = card^{\mathbf{M}}(a)$. This means that in $\mathbf{M}$ there exists some bijection f between κ and a but no bijection between a and any ordinal $\alpha < \kappa$. By absoluteness, in V, f is also a bijection between κ and a. But it may be the case that there exists (in V) some bijection between some $\lambda < \kappa$ and a. Therefore we have $card \leq card^{\mathbf{M}}(a)$. □

Solution of exercise 2:

From Exercise 3 of Sheet 7, we already know that $V_{\omega+\omega} \models ZFC - Repl$ (ZC is simply another notation for $ZFC - Repl$). It remains to check that $V_{\omega+\omega} \models \neg Repl$ i.e. we need to find some formula such that $V_{\omega+\omega}$ satisfies that it is functional on a subset A of $V_{\omega+\omega}$ but $V_{\omega+\omega}$ contains no set containing the range of this functional on A.

We consider the following formula:

$$\phi(n, \alpha) := \exists f \underbrace{\left(n \in \omega \ \wedge \ \alpha \in \mathbf{On} \ \wedge \ f \ : \ \omega + n \overset{isom.}{\longleftrightarrow} \alpha\right)}_{\psi(f,n,\alpha)}$$

. One can easily show that $\psi(f, n, \alpha)$ is absolute for $V_{\omega+\omega}$, i.e.

$$\forall f \in V_{\omega+\omega} \ \forall n \in V_{\omega+\omega} \ \forall \alpha \in V_{\omega+\omega} \ \ \left(\psi(f, n, \alpha) \ \longleftrightarrow \ \psi(f, n, \alpha)^{V_{\omega+\omega}}\right).$$

Since for all integer n, $\omega + n \in V_{\omega+\omega}$ and $id \ : \ \omega + n \longleftrightarrow \omega + n \in V_{\omega+\omega}$ hold, one has

$$\forall n \in \omega \ \exists! \alpha \in V_{\omega+\omega} \ \exists f \in V_{\omega+\omega} \ \ \psi(f, n, \alpha),$$

i.e.

$$\forall n \in \omega \ \exists! \alpha \in V_{\omega+\omega} \ \exists f \in V_{\omega+\omega} \ \ \psi^{V_{\omega+\omega}}(f, n, \alpha),$$

i.e.

$$\forall n \in \omega \ \exists! \alpha \in V_{\omega+\omega} \ \ \phi^{V_{\omega+\omega}}(n, \alpha).$$

Hence $\phi(n, \alpha)$ is functional on $V_{\omega+\omega}$. Now if $Repl^{V_{\omega+\omega}}$, then there exists some $Y \in V_{\omega+\omega}$ such that

$$Y = \{\alpha \mid n \in \omega \ \wedge \ \phi(n, \alpha)\} = \{\omega + n \mid n \in \omega\}.$$

Since $\omega \in V_{\omega+\omega}$ and $union^{V_{\omega+\omega}}$ we come to the following contradiction:

$$Y \cup \omega = \omega + \omega \in V_{\omega+\omega}.$$

Solution of exercise 3: Let us prove the following lemma, which is interesting in its own right:

Lemma (*ZFC*)**.** *Let κ be strongly inaccessible. If $x \subseteq V_\kappa$, then $x \in V_\kappa$ iff $|x| < \kappa$.*

Proof. For the first direction, it is enough to notice that for all $\alpha < \kappa$, $|V_\alpha| < \kappa$. Indeed, if $x \in V_\kappa$ then $x \in V_{\alpha+1}$ for a certain $\alpha < \kappa$ and therefore $x \subseteq V_\alpha$ and therefore $|x| \leq |V_\alpha| < \kappa$. Let us show therefore that $|V_\alpha| < \kappa$ for all $\alpha < \kappa$ by induction on α. Clearly $|V_0| < \kappa$. If for $\alpha < \kappa$ we have $|V_\alpha| < \kappa$, then $|V_{\alpha+1}| = |\mathcal{P}(V_\alpha)| = 2^{|V_\alpha|} < \kappa$ since κ is strongly limit. Suppose now that $\gamma < \kappa$ is limit and that for all $\alpha < \gamma$ we have $|V_\alpha| < \kappa$. Consider the function $f : \gamma \to \kappa$ defined by $f(\alpha) = |V_\alpha|$. By the regularity of κ, $\sup f[\gamma] < \kappa$. Notice that since f is injective $\gamma \leq \sup f[\gamma]$ and also $|V_\alpha| \leq \sup f[\gamma]$ for all $\alpha < \gamma$. We have therefore by the Exercise 2 of Sheet 5 that:

$$|V_\gamma| = \Big|\bigcup_{\alpha<\gamma} V_\alpha\Big| \leq \sup_{\alpha<\gamma} |V_\alpha| < \kappa.$$

For the opposite direction, suppose that $x \subseteq V_\kappa$ and $|x| < \kappa$. Fix a bijection $h : |x| \to x$ and define $g : |x| \to \kappa$ as $g(\xi) = \mathrm{rank}(h(\xi))$. By regularity of κ, g is not cofinal and therefore there exists an $\alpha < \kappa$ such that $\{\mathrm{rank}(y) \mid y \in x\} \subseteq \alpha$. Therefore, $x \in V_{\alpha+1} \subseteq V_\kappa$. □

We now show that $V_\kappa \models ZFC$. By Exercise 3 of Sheet 7, since $\kappa > \omega$ is limit, we have that $V_\kappa \models ZC$ and it is therefore enough to show that V_κ satisfies the axiom schema of replacement. Therefore, let $\varphi(x, y, \vec{c})$ be a formula and $a, \vec{c} \in V_\kappa$ such that $ZFC \vdash (\forall x \in a\ \exists! y \varphi(x, y, \vec{c}))^{V_\kappa}$, i.e. $ZFC \vdash \forall x \in a\ \exists! y \in V_\kappa\ \varphi(x, y, \vec{c})^{V_\kappa}$. We have therefore that $ZFC \vdash \exists f(f$ is a function of a in $V_\kappa \wedge \forall x \in a, \varphi(x, f(x), \vec{c})^{V_\kappa})$ or, in other words, φ defines a function $f : a \to V_\kappa$. It is therefore enough to show that if $a \in V_\kappa$ and $f : a \to V_\kappa$ is a function (of V), then $f \in V_\kappa$. To see this, notice first of all that since $a \in V_\kappa$ we have $a \subseteq V_\kappa$ and therefore by the previous lemma $|a| < \kappa$. Now $|f[a]| \leq |a| < \kappa$ and $f[a] \subseteq V_\kappa$ and therefore, again by the previous lemma, $f[a] \in V_\kappa$. It follows that V_κ satisfies replacement. We have thus:

$$ZFC + \exists\kappa\ \kappa \text{ is strongly inaccessible} \vdash \mathrm{Con}(ZFC).$$

We now show:

Theorem. *$Con(ZFC) \to Con(ZFC + \neg\exists\kappa(\kappa$ strongly inaccessible$))$,*
i.e. $Con(ZFC) \to$ "$ZFC \nvdash \exists\kappa(\kappa$ strongly inaccessible$)$".

Let $\mathrm{SI}(\kappa)$ be the formula expressing "κ is strongly inaccessible". We use the following result[1]:

[1] This lemma comes from *www.cis.upenn.edu/~byorgey/settheory/*, April 28, 2012.

Lemma 10.5 (Absoluteness of SI)**.** *If λ is a limit ordinal and $\kappa \in V_\lambda$, then $SI(\kappa) \iff V_\lambda \models SI(\kappa)$.*

Proof. Unfolding the definition of SI, it suffices to show each of the following.

- $\mathrm{ord}(\kappa) \iff V_\lambda \models \mathrm{ord}(\kappa)$. Since we have the Axiom of Regularity, $\mathrm{ord}(\kappa)$ simply reduces to the statement that κ is a transitive linear order, both of which are Δ_0 conditions.

- $\mathrm{card}(\kappa) \iff V_\lambda \models \mathrm{card}(\kappa)$. Recall that $\mathrm{card}(\kappa)$ holds iff there is no f for which there exists some $\beta < \kappa$ such that $f : \beta \xrightarrow[\text{onto}]{\text{1-1}} \kappa$.

  First, suppose $\mathrm{card}(\kappa)$, that is, there is no bijection in the universe between κ and some $\beta < \kappa$. If there is no such bijection in the universe, there isn't one in V_λ either, since the notion of being a bijection between β and κ is Δ_0.

  Now, suppose $V_\lambda \models \mathrm{card}(\kappa)$, and suppose by way of contradiction that there is some f in the universe which is a bijection between κ and some $\beta < \kappa$. Note that $f \subseteq \beta \times \kappa \subseteq \mathcal{P}(\mathcal{P}(\beta \cup \kappa))$, so its rank is at most two greater than the rank of κ. But $\kappa \in V_\lambda$, and since λ is a limit ordinal, $\kappa \in V_\alpha$ for some $\alpha < \lambda$, and hence $f \in V_{\alpha+2} \subseteq V_\lambda$, which is a contradiction.

- $\mathrm{cf}(\kappa) = \kappa \iff V_\lambda \models \mathrm{cf}(\kappa) = \kappa$. We can also restate $\mathrm{cf}(\kappa) = \kappa$ as the fact that there is no ordinal $\alpha < \kappa$ for which there exists a cofinal map $f : \alpha \to \kappa$.

  ($\Longrightarrow$) Suppose there is no ordinal $\alpha < \kappa$ in the universe for which there exists a cofinal map $f : \alpha \to \kappa$. Then there is no such ordinal in V_λ, either, since the notion of being a cofinal map from $\alpha \to \kappa$ is absolute for V_λ (this is because $\alpha, \kappa \in V_\lambda$; the notion of being a functional relation from α to κ is absolute for V_λ; and the predicate defining what it means to be a *cofinal* map only has to talk about union, which lowers rank).

  ($\Longleftarrow$) Suppose that $V_\lambda \models \mathrm{cf}(\kappa) = \kappa$, and suppose by way of contradiction that there is some $\alpha < \kappa$ and a cofinal map $f : \alpha \to \kappa$. Clearly $\alpha \in V_\lambda$. It is also easy to see that $f \in V_\lambda$ by the same argument as in the previous case.

- κ is a strong limit cardinal $\iff V_\lambda \models \kappa$ is a strong limit cardinal.

  First, suppose κ is a strong limit cardinal. This means that $2^\iota < \kappa$ for every cardinal $\iota < \kappa$, which is the case if and only if, for every $\iota < \kappa$, there is an injection $f : \mathcal{P}(\iota) \xrightarrow{\text{1-1}} \kappa$. By the usual rank argument, $f \in V_\lambda$.

  Now suppose $V_\lambda \models$ (κ is a strong limit cardinal), which means that for every cardinal $\iota < \kappa$, there is some $f \in V_\lambda$ such that $f : \mathcal{P}(\iota) \xrightarrow{\text{1-1}} \kappa$. But then for each ι, that f is evidence in the universe that $2^\iota < \kappa$; hence κ is a strong limit cardinal.

- κ is uncountable $\iff V_\lambda \models \kappa$ is uncountable.

  First, suppose κ is uncountable; then there does not exist any function $f : \kappa \xrightarrow{\text{1-1}} \omega$. Then in particular, there does not exist any such function in V_λ, since being an injection from κ into ω is absolute for V_λ.

  Now, suppose $V_\lambda \models \kappa$ is uncountable. By way of contradiction, suppose there is some f in the universe with $f : \kappa \xrightarrow{\text{1-1}} \omega$. By an easy rank argument (noting that κ uncountable implies $\lambda > \omega$), $f \in V_\lambda$.

□

Proof of the Theorem. Consider the following class:

$$\mathbf{M} = \{x \mid \forall \kappa (\mathrm{SI}(\kappa) \to x \in V_\kappa)\}.$$

From a semantic standpoint, $\mathbf{M}$ could be V_κ, for κ the smallest strongly inaccessible cardinal, as well as the whole class V, depending on whether there exists or not a strongly inaccessible cardinal. However, in either case, $\mathbf{M} \models ZFC + \neg\exists\kappa \mathrm{SI}(\kappa)$. Actually:

- If $\neg\exists\kappa\ \mathrm{SI}(\kappa)$: in this case $\mathbf{M} = V$ and therefore $\mathbf{M} \models \neg\exists\kappa\ \mathrm{SI}(\kappa)$.
- If $\exists\kappa\ \mathrm{SI}(\kappa)$: in this case $\mathbf{M} = V_\kappa$ for κ the smallest strongly inaccessible cardinal. In this case, by the first part of this exercise, $V_\kappa \models ZFC$. Moreover, by the lemma, $\mathrm{SI}(\alpha)$ is absolute for V_κ and therefore $\forall\alpha \in V_\kappa(\mathrm{SI}(\alpha) \leftrightarrow \mathrm{SI}(\alpha)^{V_\kappa})$. By minimality of κ, $\forall\alpha \in V_\kappa\ \neg\mathrm{SI}(\alpha)$ and therefore $\forall\alpha \in V_\kappa\ \neg\mathrm{SI}(\alpha)^{V_\kappa}$, in other words $(\neg\exists\alpha\ \mathrm{SI}(\alpha))^{V_\kappa}$, or equivalently $V_\kappa \models \neg\exists\alpha\ \mathrm{SI}(\alpha)$.

Therefore, $\mathbf{M} \models \neg\exists\alpha \mathrm{SI}(\alpha)$ as wanted. □

Finally notice the following fact.

Theorem. *If ZFC is consistent, then it is not possible to formalize the following proof in ZFC:*

$$Con(ZFC) \to Con(ZFC + \exists\kappa\, \mathrm{SI}(\kappa)).$$

Proof. Suppose towards contradiction that ZFC is consistent and that $ZFC \vdash \mathrm{Con}(ZFC) \to \mathrm{Con}(ZFC + \exists\kappa\, \mathrm{SI}(\kappa))$. We have then in particular that $ZFC + \exists\kappa\, \mathrm{SI}(\kappa) \vdash \mathrm{Con}(ZFC) \to \mathrm{Con}(ZFC + \exists\kappa\, \mathrm{SI}(\kappa))$ and therefore by the first part of this exercise, which can in fact be formalized as:

$$ZFC \vdash \text{“}ZFC + \exists\kappa\, \mathrm{SI}(\kappa) \vdash \mathrm{Con}(ZFC)\text{”},$$

we would have that $ZFC + \exists\kappa \mathrm{SI}(\kappa) \vdash \mathrm{Con}(ZFC + \exists\kappa \mathrm{SI}(\kappa))$. Once formalized, it yieds:

$$ZFC \vdash \text{“}ZFC + \exists\kappa \mathrm{SI}(\kappa) \vdash \mathrm{Con}(ZFC + \exists\kappa \mathrm{SI}(\kappa))\text{”}$$

By Gödel's second incompleteness theorem (formalized in ZFC) we obtain that $ZFC \vdash \neg\mathrm{Con}(ZFC + \exists\kappa\, \mathrm{SI}(\kappa))$. Therefore, by our second hypothesis, we have that $ZFC \vdash \neg\mathrm{Con}(ZFC)$, contradicting our hypothesis on the consistency of ZFC. □