Solution Sheet n°6

Solution of exercise 1:

Let φ be a bijection between $\mathbb{N}$ and the set of finite parts of $\mathbb{N}$, and let $\in_\varphi \subseteq \mathbb{N} \times \mathbb{N}$ be the binary relation defined by $x \in_\varphi y$ iff $x \in \varphi(y)$.

1. The proof presented in exercise 3 of the first series adapts easily to this case. In that exercise, the only time a specific property of the bijection was used was actually to prove that the axiom of foundation held. We obtain thus that the structure $U_\varphi = (\mathbb{N}, \in_\varphi)$ is a model of ZFC^-_{fin}.

2. If the bijection $\varphi : \mathbb{N} \to \mathcal{P}_{\text{fin}}(\mathbb{N})$ is such that for all $p, q \in \mathbb{N}$, $p \in \varphi(q)$ implies $p < q$, then the argument given for the axiom of foundation in exercise 3 of first series easily adapts to this case.

3. We can, for example, slightly modify the bijection of exercise 3 of the first series. The first values of this bijection are the following:

$$\begin{aligned} 0 &\mapsto [0] = \emptyset \\ 1 &\mapsto [1] = \{0\} \\ 2 &\mapsto [2] = \{1\} \\ 3 &\mapsto [3] = \{0, 1\} \\ 4 &\mapsto [4] = \{2\} \\ &\vdots \end{aligned}$$

   We then define $\varphi : \mathbb{N} \to \mathcal{P}_{\text{fin}}$ by letting $\varphi(1) = \{1\}$, $\varphi(2) = \{0\}$ and $\varphi(p) = [p]$ if $p \neq 1, 2$. Notice that this bijection is such that $U_\varphi = (\mathbb{N}, \in_\varphi)$ is a model of ZFC^-_{fin} by point 1, but it does not satisfy the condition of point 2, since $1 \in \varphi(1)$ or $1 \not< 1$.

   Finally let us show that $U_\varphi = (\mathbb{N}, \in_\varphi)$ satisfies the negation of the axiom of foundation. We have that 1 is not $\in_\varphi$-empty, but the only $\in_\varphi$-element of 1 is 1. Moreover, $1 \in_\varphi 1 \in_\varphi 1$. Indeed we have the following infinite decreasing chain for $\in_\varphi$

$$1 \ni_\varphi 1 \ni_\varphi 1 \ni_\varphi 1 \ni_\varphi 1 \ni_\varphi 1 \cdots$$

   Therefore the axiom of foundation is not verified in U_φ and therefore U_φ is a model of $ZFC^-_{\text{fin}} \cup \{\neg AF\}$.

Solution of exercise 2:

1. For example, let $N = V \setminus \{1\}$. Consider the sets $x = \{0,1,2\}$ and $y = \{0,2\}$, then:

$$\forall z \in \mathbf{N}(z \in x \longleftrightarrow z \in y)$$

   but $x \neq y$.

2. 1. <u>"$\mathbf{M} \models$ Ext."</u>

      We need to prove that $\text{Ext.}^{\mathbf{M}}$ holds, where $\text{Ext.}^{\mathbf{M}}$ is the formula:

$$\forall x, y \in \mathbf{M}\Big(\forall z \in \mathbf{M}(z \in x \longleftrightarrow z \in y) \longrightarrow x = y\Big).$$

So formally we need to prove:

$$ZF \vdash \text{“}\mathbf{M} \text{ transitive}'' \longrightarrow \forall x, y \in \mathbf{M}\Big(\forall z \in \mathbf{M}(z \in x \longleftrightarrow z \in y) \longrightarrow x = y\Big).$$

Since $\mathbf{M}$ is transitive, if $z \in x \in \mathbf{M}$ then $z \in \mathbf{M}$. Therefore we have:

$$\text{“}\mathbf{M} \text{ transitive}'' \longrightarrow \forall x, y \in \mathbf{M}\Big(\forall z \in \mathbf{M}(z \in x \longleftrightarrow z \in y) \longrightarrow x = y\Big)$$

$$\Longleftrightarrow$$

$$\text{“}\mathbf{M} \text{ transitive}'' \longrightarrow \forall x, y \in \mathbf{M}\Big(\forall z(z \in x \longleftrightarrow z \in y) \longrightarrow x = y\Big).$$

$(\Rightarrow)$ is by transitivity, and $(\Leftarrow)$ is trivial.
Finally, $\forall x, y \in \mathbf{M}\Big(\forall z(z \in x \longleftrightarrow z \in y) \longrightarrow x = y\Big)$ holds by Extensionality.

2. If $\mathbf{M} \subseteq \mathbf{WF}$, then “$\mathbf{M} \models$ Foundation”.
“Foundation$^{\mathbf{M}}$” is the formula:

$$\forall x \in \mathbf{M}\Big(\exists y \in \mathbf{M}(y \in x) \to \exists y \in \mathbf{M}\big(y \in x \land \neg\exists z \in \mathbf{M}(z \in x \land z \in y)\big)\Big).$$

Any y of minimal rank in $x \cap \mathbf{M}$ satisfies the above formula.

3. “$\mathbf{M} \models$ Pairing” iff $\forall a, b \in \mathbf{M}\ \exists c \in \mathbf{M}\ \{a, b\} \subseteq c$.
“Pairing$^{\mathbf{M}}$” is precisely the formula:

$$\forall x \in \mathbf{M}\ \forall y \in \mathbf{M}\ \exists z \in \mathbf{M}\ \ (x \in z \land y \in z),$$

which is

$$\forall x \in \mathbf{M}\ \forall y \in \mathbf{M}\ \exists z \in \mathbf{M}\ \ (\{x, y\} \subseteq z).$$

4. “$\mathbf{M} \models$ Union” iff $\forall a \in \mathbf{M}\ \exists b \in \mathbf{M}\ \ \bigcup a \subseteq b$.
“Union$^{\mathbf{M}}$” is precisely the formula:

$$\forall a \in \mathbf{M}\ \exists b \in \mathbf{M}\ \forall x \in \mathbf{M}\ \forall y \in \mathbf{M}\ (x \in y \land y \in a \to x \in b).$$

Assuming $\mathbf{M}$ is transitive we obtain:

$$\forall a \in \mathbf{M}\ \exists b \in \mathbf{M}\ \forall x \in \mathbf{M}\ \forall y \in \mathbf{M}\ (x \in y \land y \in a \to x \in b)$$

$$\Longleftrightarrow^{1}$$

$$\forall a \in \mathbf{M}\ \exists b \in \mathbf{M}\ \forall x\ \forall y\ (x \in y \land y \in a \to x \in b)$$

$$\Longleftrightarrow$$

$$\forall a \in \mathbf{M}\ \exists b \in \mathbf{M}\ \ \bigcup a \subseteq b.$$

[1] $(\Rightarrow)$ is by transitivity, and $(\Leftarrow)$ is trivial.

5. "$\mathbf{M} \models$ PowerSet" iff $\forall a \in \mathbf{M}\ \exists c \in \mathbf{M}\ (\mathcal{P}(a) \cap \mathbf{M} \subseteq c)$.
   "PowerSet$^{\mathbf{M}}$" is the formula:

$$\forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \forall z \in \mathbf{M}\ \ \big(\forall u \in \mathbf{M}\ (u \in z \to u \in x) \to z \in y\big).$$

   Assuming $\mathbf{M}$ is transitive we have:

$$\forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \forall z \in \mathbf{M}\ \ \big(\forall u \in \mathbf{M}\ (u \in z \to u \in x) \to z \in y\big)$$

$$\Longleftrightarrow$$ [2]

$$\forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \forall z \in \mathbf{M}\ \ \big(\forall u\ (u \in z \to u \in x) \to z \in y\big)$$

$$\Longleftrightarrow$$

$$\forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \forall z \in \mathbf{M}\ \ \big(z \subseteq x \to z \in y\big)$$

$$\Longleftrightarrow$$

$$\forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \ \big(\mathcal{P}(x) \cap \mathbf{M} \subseteq y\big).$$

6. "$\mathbf{M} \models$ Comp.Schema" iff for every formula $\varphi := \varphi(x, z, \overline{w})$ whose free variables are among $x, z, w_1, ..., w_n$, we have

$$\forall z \in \mathbf{M}\ \forall w_1 \in \mathbf{M}\ ...\forall w_n \in \mathbf{M}\ \ \{x \in z : \varphi^{\mathbf{M}}(x, z, \overline{w})\} \in \mathbf{M}$$

   "Comp.Schema$^{\mathbf{M}}$" is the class of formulas of the form:

$$\forall z \in \mathbf{M}\ \forall w_1 \in \mathbf{M}\ ...\forall w_n \in \mathbf{M}\ \exists y \in \mathbf{M}\ \forall x \in \mathbf{M}\ \big(x \in y \leftrightarrow (x \in z \land \varphi^{\mathbf{M}})\big),$$

   where $\varphi := \varphi(x, z, \overline{w})$ is any formula with free variables among $x, z, w_1, ..., w_n$.
   Assuming $\mathbf{M}$ is transitive we obtain:

$$\forall z \in \mathbf{M}\ \forall w_1 \in \mathbf{M}\ ...\forall w_n \in \mathbf{M}\ \exists y \in \mathbf{M}\ \forall x \in \mathbf{M}\ \big(x \in y \leftrightarrow (x \in z \land \varphi^{\mathbf{M}})\big)$$

$$\Longleftrightarrow$$ [3]

$$\forall z \in \mathbf{M}\ \forall w_1 \in \mathbf{M}\ ...\forall w_n \in \mathbf{M}\ \exists y \in \mathbf{M}\ \forall x\ \big(x \in y \leftrightarrow (x \in z \land \varphi^{\mathbf{M}})\big)$$

$$\Longleftrightarrow$$

$$\forall z \in \mathbf{M}\ \forall w_1 \in \mathbf{M}\ ...\forall w_n \in \mathbf{M}\ \exists y \in \mathbf{M}\ \ y = \big\{x \in z \bigm| \varphi^{\mathbf{M}}\big\}.$$

7. By previous case and transitivity of $\mathbf{M}$, $\{y \mid \exists x \in A\ \varphi(x, y, A, \vec{c})\}^{\mathbf{M}} = \{y \mid (\exists x \in A\ \varphi(x, y, A, \vec{c}))^{\mathbf{M}}\} = \{y \mid \exists x \in A\ \varphi(x, y, A, \vec{c})^{\mathbf{M}}\}$. Moreover, by transitivity of $\mathbf{M}$ the relativisation of the inclusion to $\mathbf{M}$ is equivalent to the inclusion in V, which yields the result.

[2] ($\Rightarrow$) is by transitivity, and ($\Leftarrow$) is trivial.
[3] ($\Rightarrow$) is by transitivity, and ($\Leftarrow$) is trivial.

Solution of exercise 3:

1. Working in ZF^{-}, show that "$\mathbf{V}_\omega \models ZF - Inf + \neg Inf$".

   The set $\mathbf{V}_\omega$ is transitive. Moreover, for $a, b \in \mathbf{V}_\omega$ we have that $\{a, b\}$, $\bigcup a$, $\mathcal{P}(a)$ all belong to $\mathbf{V}_\omega$. Therefore, by Exercise 2, the axioms of extensionality, foundation, pairing, union and powerset all hold in $\mathbf{V}_\omega$. Moreover, since for all $a \in \mathbf{V}_\omega$ we have that $\mathcal{P}(a) \in \mathbf{V}_\omega$ it follows, in particular, that for every formula φ, $\{x \in a \mid \varphi\} \in \mathbf{V}_\omega$.

   For the replacement axiom, let $a \in \mathbf{V}_\omega$ and $F(x, y, \overrightarrow{z})$ be such that:

   $$\forall x \in a \ \exists! y \in \mathbf{V}_\omega \quad F^{\mathbf{V}_\omega}(x, y, \overrightarrow{z}).$$

   Consider $b = \{y \in \mathbf{V}_\omega \mid \exists x \in a \ \ F^{\mathbf{V}_\omega}(x, y, \overrightarrow{z})\}$; since $a \in \mathbf{V}_\omega$, it follows that $a \in \mathbf{V}_n$ for an integer n and therefore a is finite. Therefore, since by construction $b \subseteq \mathbf{V}_\omega$ and a surjects on b, it follows that b is also finite and belongs therefore to $\mathbf{V}_m$ for some integer m.

   To show that $\neg Inf^{\mathbf{V}_\omega}$ holds we suppose towards contradiction that $Inf^{\mathbf{V}_\omega}$ holds, i.e.:

   $$\exists x \in \mathbf{V}_\omega \ \big(\exists y \in \mathbf{V}_\omega \ (y \in x \wedge \forall z \in \mathbf{V}_\omega \ \ z \notin y) \wedge \forall z \in \mathbf{V}_\omega \ (z \in x \to z \cup \{z\} \in x)\big)$$

   $$\iff^{4}$$

   $$\exists x \in \mathbf{V}_\omega \ \big(\exists y \in \mathbf{V}_\omega \ (y \in x \wedge \forall z \ \ z \notin y) \wedge \forall z \ (z \in x \to z \cup \{z\} \in x)\big)$$

   $$\iff$$

   $$\exists x \in \mathbf{V}_\omega \ \big(\emptyset \in x \wedge \forall z \ (z \in x \to z \cup \{z\} \in x)\big),$$

   from which it would follow that there exists an element of $\mathbf{V}_\omega$ which contains all the integer. By comprehension $\omega \in \mathbf{V}_\omega$, a contradiction.

2. It follows from the proposition in exercise 2 and from results which have been presented during class that $\mathbf{WF}$ satisfies $ZF - Inf$. More precisely, we have that $ZF^{-} \vdash (\phi)^{\mathbf{WF}}$ for every formula of $ZF - Inf$. It remains to prove that $ZF^{-} \vdash (Inf)^{\mathbf{WF}}$. The idea is that ω satisfies the statement of the axiom of infinity relativised to $\mathbf{WF}$. Another observation is that the notion of empty set and the notion of successor are the same in V and in $\mathbf{WF}$. More precisely, we have that:

   $$\begin{aligned} ZF^{-} \vdash \quad & (\exists x(\emptyset \in x \wedge \forall y \in x \ S(y) \in x))^{\mathbf{WF}} \\ & \leftrightarrow \exists x \in \mathbf{WF} \ (\forall z((\forall w \in z \ w \neq w) \to z \in x) \wedge \forall y \in x \ \bigcup\{y, \{y\}\} \in x)^{\mathbf{WF}} \\ & \leftrightarrow \exists x \in \mathbf{WF} \ (\forall z((\forall w \in z \ w \neq w) \to z \in x) \wedge \forall y \in x \ \bigcup\{y, \{y\}\} \in x) \\ & \leftrightarrow \exists x \in \mathbf{WF} \ (\emptyset \in x \wedge \forall y \in x \ S(y) \in x). \end{aligned}$$

   The second equivalence is obtained using the fact that $\mathbf{WF}$ is a transitive class satisfying $ZF - Inf$. The last formula is satisfied by letting $x = \omega$.

3. By arguments similar to those in point 1, we have that $\mathbf{V}_\lambda$ is a model of Z.

   For the axiom of choice, it is enough to find, for each $a \in \mathbf{V}_\lambda$, a well order on a which is in $\mathbf{V}_\lambda$. Now, if $a \in \mathbf{V}_\lambda$, there exists an ordinal $\xi < \lambda$ such that $a \in \mathbf{V}_\xi$. Moreover, since AC holds, there exists a well order $<_a \ \subseteq a \times a$ on a. Therefore $<_a \in \mathbf{V}_{\xi+3}$ and thus $<_a \in \mathbf{V}_\lambda$. Moreover, the formula $\varphi(a, <_a) :=$ "$<_a$ well orders a" verifies $\varphi(a, <_a) \leftrightarrow \varphi(a, <_a)^{\mathbf{V}_\lambda}$. Thus $AC^{\mathbf{V}_\lambda}$.

[4] $(\Rightarrow)$ by transitivity of $\mathbf{V}_\omega$, $(\Leftarrow)$ is trivial.