Solution Sheet n°5

Solution of exercise 1:

1. It follows from:

$$x \in \mathbf{W}(\alpha+1) \leftrightarrow x \in \mathcal{P}\big(\mathbf{W}(\alpha)\big) \leftrightarrow x \subseteq \mathbf{W}(\alpha).$$

2. In order to show that each of these sets is in **WF**, one can either show that there is no infinite strictly $\in$-decreasing chain (see Corollary 144 of the Lecture Notes), or use the tree representation of the well-founded sets (see Exercice 2 of this Exercice Sheet).

   (a) $rk(\{x,y\}) = \max\{rk(x),rk(y)\} + 1$;

   (b) Since $(x,y) = \{\{x\},\{x,y\}\}$, we get $rk\big((x,y)\big) = \max\{rk(x),rk(y)\} + 2$;

   (c) Since $\langle x,y\rangle = \{(0,x),(1,y)\}$, we get $rk(\langle x,y\rangle) = \max\{rk(x),rk(y),1\} + 3$;

   (d) Since $\langle x,y,x\rangle = \{(0,x),(1,y),(2,x)\}$, we get $rk(\langle x,y,x\rangle) = \max\{rk(x),rk(y),2\} + 3$;

   (e) By the point 1. of this exercice, we have $x \subseteq \mathbf{W}(rk(x))$, thus, for any $y \in x$, we have $y \in \mathbf{W}(rk(x))$. Thus, $y \subseteq \bigcup_{\beta<rk(x)} \mathbf{W}(\beta)$. This implies that, for any $z \in y \in x$, we have $z \in \bigcup_{\beta<rk(x)} \mathbf{W}(\beta)$. We finally obtain $rk\big(\bigcup x\big) \leq rk(x)$ with equality exactly when $rk(x)$ is limit;

   (f) By the point 1. of this exercice, we get $rk\big(\mathcal{P}(x)\big) = rk(x) + 1$;

   (g) By (a) and (e), and since $x \cup y = \bigcup\{x,y\}$, we get $rk(x\cup y) = \max\{rk(x),rk(y)\}$;

   (h) By (b), $rk(x \times y) = \max\{rk(x),rk(y)\} + 2$;

   (i) Since $f \subseteq x \times y$, by (h), we obtain $rk(f) \leq \max\{rk(x),rk(y)\} + 2$;

   (j) By (i), $rk({}^{x}y) \leq \max\{rk(x),rk(y)\} + 3$;

   (k) By the point 1. of this exercice, we get $rk\big(\mathbf{W}(\alpha)\big) = \alpha$;

   (l) Since $R \subseteq \mathbf{W}(\alpha) \times \mathbf{W}(\alpha)$, (h) implies that $rk(R) \leq \alpha + 2$.

3. Consider $x = \{2k : k \in \omega\}$ and $y = \{\omega\}$. On the one hand, we have $\bigcup x = \bigcup y = \omega$, thus $rk\big(\bigcup x\big) = rk\big(\bigcup x\big) = rk(\omega) = \omega$. But, on the other hand, we have $rk(x) = \omega$ and $rk(y) = \omega + 1$.

4. (a) It follows from the second point of this exercice.

   (b) $rk(\mathbb{N}) = \omega$, and $rk(n) = n$ for any $n \in \mathbb{N}$;
   $rk(\mathbb{Z}) = \omega + 1$, and $rk(z) \leq \omega$ for any $z \in \mathbb{Z}$;
   $rk(\mathbb{Q}) = \omega + 4$, and $rk(q) \leq \omega + 3$ for any $q \in \mathbb{Q}$;
   $rk(\mathbb{R}) = \omega + 5$, and $rk(r) \leq \omega + 4$ for any $r \in \mathbb{R}$.

5. It suffices to consider

$$\mathbb{Q} = \omega \cup \{\langle k,l,m\rangle : k,l,m \in \mathbb{N},\ k \in \{0,1\},\ gcd(l,m) = 1,\ (l = 0 \to n \geq 2)\}.$$

   Indeed, the second point of this exercice implies that $rk(q) < \omega$ for any $q \in \mathbb{Q}$, and that $rk(\mathbb{Q}) = \omega$.

   If we define (as in Exercice Sheet 2) the real numbers $\mathbb{R}$ as Dedekind cuts from this new definition of $\mathbb{Q}$, we obtain $rk(\mathbb{R}) = \omega + 1$ and $rk(r) = \omega$ for any $r \in \mathbb{R}$.

It is not possible to define $\mathbb{R}$ at a lower level. Indeed, $\mathbb{R}$ is uncountable and there are countably many elements in $\mathbf{W}(\omega)$.

Solution of exercise 2:

1. Define the well-founded tree $T_{\bigcup x}$ as:

$$s \in T_{\bigcup x} \subseteq A^{<\omega} \leftrightarrow \begin{cases} s = \varnothing \text{ or,} \\ \exists a \in A \text{ such that } a^\frown s \in T_x. \end{cases}$$

It is easy to check that $T_{\bigcup x}$ is a well-founded tree. We verify that the set y represented by the tree $T_{\bigcup x}$ is equal to $\bigcup x$. For any $s \in T$, let us denote by T^s the tree T that starts from $s \in T$. Observe that T_z represents $z \in \mathbf{WF}$ if and only if

$$z = \left\{u \in \mathbf{WF} : u \text{ is represented by } T_z^s, \text{ where } s \text{ is a child of } \varnothing \text{ in } T_z\right\}.$$

$y \subseteq \bigcup x$: Suppose that $z \in y$, then there exists a child s of $\varnothing$ in $T_{\bigcup x}$ such that $T^s_{\bigcup x}$ represents the set z. By definition of $T_{\bigcup x}$, there exists a child t of $\varnothing$ such that $t^\frown s \in T_x$. Thus, s is a child of $\varnothing$ in T_x^t (which represents a set u), which means that $z \in u$, and t is a child of $\varnothing$ in T_x, which means $u \in x$. Thus, we get $z \in u \in x$, which implies $z \in \bigcup x$.

$\bigcup x \subseteq y$: Suppose that $z \in \bigcup x$, then there exists u such that $z \in u \in x$. If u is represented by T_x^s with s a child of $\varnothing$, then z is represented by $\left(T_x^s\right)^t$ with t a child of $\varnothing$ in T_x^s. Since $\left(T_x^s\right)^t = T_x^{s^\frown t}$, we get that $s^\frown t \in T_x$, which implies that t is a child of $\varnothing$ in $T_{\bigcup x}$, thus $z \in y$.

2. By the previous point, it is easy to define a well-founded tree T_n on A that represents $\bigcup^n x$ for any $n \in \omega$. Moreover, it is also easy to define a well-founded tree T' that represents $\left\{\bigcup^n x : n \in \omega\right\}$. Indeed, it suffices to let $s \in T' \subseteq (A \cup \omega)^{<\omega}$ if and only if $s = \langle n \rangle^\frown t$ and $t \in T_n$. To conclude, it suffices to construct the well-founded tree $T \subseteq (A \cup \omega)^{<\omega}$ that represents $tc(x) = \bigcup \left\{\bigcup^n x : n \in \omega\right\}$ using the previous point.

Solution of exercise 3:

1. Set $\beta_0 = \aleph_0$ and $\beta_{n+1} = \aleph_{\beta_n}$ for any $n \in \omega$, and consider the cardinal $\kappa = \sup_{n\in\omega} \beta_n$. Then, we have:

$$\aleph_\kappa = \sup_{\beta<\kappa} \aleph_\beta = \sup_{n\in\omega} \aleph_{\beta_n} = \sup_{n\in\omega} \beta_{n+1} = \kappa.$$

2. Let κ be a strong inaccessible cardinal. By definition, we have $\aleph_0 < \kappa$, $\mathrm{cof}(\kappa) = \kappa$ and for all $\beta < \kappa$, we have $2^\beta < \kappa$. We proceed by transfinite induction to show that $\aleph_\beta < \kappa$ for any $\beta < \kappa$:
   - $\aleph_0 < \kappa$;
   - $\aleph_{\beta+1} \leq 2^{\aleph_\beta} < \kappa$;
   - $\aleph_\lambda = \sup_{\beta<\lambda} \aleph_\beta < \kappa$ by regularity and the fact that $\lambda < \kappa$.

   Thus, $\kappa \leq \aleph_\kappa \leq \kappa$ is a fixed point.

Solution of exercise 4:

1. $\exists x \ (\varphi_{\mathbf{C}}(x) \wedge x = x)$;
2. $\forall x \ \big(\varphi_{\mathbf{C}}(x) \to \exists y \ (\varphi_{\mathbf{C}}(y) \wedge x \in y)\big)$;
3. $\forall x \Bigg(\varphi_{\mathbf{C}}(x) \to \forall y \Bigg(\varphi_{\mathbf{C}}(y) \to \Big(\forall z \big(\varphi_{\mathbf{C}}(z) \to (z \in x \leftrightarrow z \in y)\big)\Big) \to x = y\Bigg)\Bigg)$;
4. $\forall x \Bigg(\varphi_{\mathbf{C}}(x) \to \exists y \Bigg(\varphi_{\mathbf{C}}(y) \wedge \forall z \Big(\varphi_{\mathbf{C}}(z) \to \big(\forall u(\varphi_{\mathbf{C}}(u) \to [u \in z \to u \in x]) \to z \in y\big)\Big)\Bigg)\Bigg)$.

Solution of exercise 5:

1. Each family τ_i contains the empty set and the entire space and moreover is closed by finite intersection and arbitrary union, it is therefore a topology on X_i. Also, a finite topology is always compact.

2. Notice that for each $i \in I$, the projection $p_i : \prod_{j \in I} X_j \to X_i, (x_j) \mapsto x_i$ is continuous by definition of the product. Moreover, A_i is closed in X_i, being the complement of $\{\alpha\}$. Finally, since p_i is surjective and A_i is non empty by hypothesis, we have that $C_i = p_i^{-1}(A_i)$ is a closed non empty subset of $\prod_{j \in I} X_j$.

3. Recall that the compactness of a topological space is equivalent to the statement: all families of closed sets $\mathcal{F}$ such that for each finite subfamily $G \subseteq \mathcal{F}$, we have that $\bigcap G \neq \emptyset$ admit a non empty intersection. By Tychonoff's theorem, $\prod_{j \in I} X_j$ is compact and it is therefore enough to show that for all finite subsets $J \subseteq I$ we have $\bigcap_{j \in J} C_j \neq \emptyset$. So let J be a subset of I, since J is finite, and the A_j are non empty, we can chose (without using AC!) some $a_j \in A_j$ for each $j \in J$. The element $(x)_{i \in I} \in \prod_{i \in I} X_i$, defined by:

   $$x_i = \begin{cases} a_i & \text{if } i \in J, \\ \alpha & \text{if not,} \end{cases}$$

   belongs then to $\bigcap \{C_j \mid j \in J\}$.

4. By the previous point, there exists an element $(x_i)_{i \in I} \in \bigcap \{C_j \mid j \in I\}$. This element is a function of $I \to \bigcup X_i$ such that for all $i \in I$, $x_i \in A_i$. There exists therefore a choice function for the family $(A_i)_{i \in I}$.