Solution Sheet n°4

Solution of exercise 1:

1. By definition $E_0 = E \subseteq \mathrm{cl}_{\mathcal{O}}(E)$. Let α be such that $E_\alpha \subseteq \mathrm{cl}_{\mathcal{O}}(E)$, then, for all $f \in \mathcal{O}$ and all $(x_\xi)_{\xi<\kappa} \in (E_\alpha)^\kappa$, we have $f((x_\xi)) \in \mathrm{cl}_{\mathcal{O}}(E)$ since $\mathrm{cl}_{\mathcal{O}}(E)$ is closed under $\mathcal{O}$ and therefore we have $E_{\alpha+1} \subseteq \mathrm{cl}_{\mathcal{O}}(E)$. For λ limit it is enough to notice that $\forall \xi < \lambda$, $E_\xi \subseteq \mathrm{cl}_{\mathcal{O}}(E)$, then $E_\lambda = \bigcup_{\xi<\lambda} E_\xi \subseteq \mathrm{cl}_{\mathcal{O}}(E)$.

2. Show that E_{κ^+} is closed under the operations in $\mathcal{O}$.

   Let $(x_\xi)_{\xi<\kappa} \in (E_{\kappa^+})^\kappa$. We define $f : \kappa \to \kappa^+$ by:

   $$f(\xi) = \min\{\alpha < \kappa^+ \mid x_\xi \in E_\alpha\}.$$

   Since κ^+ is regular, f is not cofinal and there exists therefore $\zeta < \kappa^+$ such that $f(\xi) \leq \zeta$ for all $\xi < \kappa$. Since the succession $(E_\alpha)_{\alpha\in\mathbf{ON}}$ is monotone with respect to inclusion, it follows that $(x_\xi)_{\xi<\kappa} \in E_\zeta$ and therefore, for all $g \in \mathcal{O}$, we have $g((x_\xi)) \in E_{\zeta+1}$. Therefore E_{κ^+} is closed under the operations of $\mathcal{O}$.

3. By point 1., we have $E_{\kappa^+} \subseteq \mathrm{cl}_{\mathcal{O}}(E)$. Moreover $E \subseteq E_{\kappa^+}$ and by 2. E_{κ^+} is closed under the operations of $\mathcal{O}$ and therefore $\mathrm{cl}_{\mathcal{O}}(E) \subseteq E_{\kappa^+}$.

4. Denote by $\mathcal{T}$ the usual topology $\mathbb{R}$. Since $\mathbb{R}$ admits a countable basis $\mathcal{B}$ of open sets, we have that $|\mathcal{T}| \leq \mathcal{B}^{\aleph_0} = \aleph_0^{\aleph_0} = 2^{\aleph_0}$. Moreover, the application $\mathbb{R}^+ \to \mathcal{T}$ defined by $x \mapsto (0,x)$ is injective and therefore $2^{\aleph_0} = |\mathbb{R}^+| \leq |\mathcal{T}|$. Thus, $|\mathcal{T}| = 2^{\aleph_0}$.

5. Denote by $c, u, i : (\mathcal{P}(\mathbb{R}))^{\aleph_0} \to \mathcal{P}(\mathbb{R})$ the operations defined by:

   $$c((A_n)_{n<\aleph_0}) = \mathbb{R} \setminus A_0;$$
   $$u((A_n)_{n<\aleph_0}) = \bigcup_{n<\aleph_0} A_n;$$
   $$i((A_n)_{n<\aleph_0}) = \bigcap_{n<\aleph_0} A_n.$$

   By definition, the set $\mathcal{B}$ of Borel subsets of $\mathbb{R}$ is the closure of $\mathcal{T}$ under the operations c, u and i. By the first part of this exercise, we have $\mathcal{B} = \bigcup_{\alpha<\aleph_1} \mathcal{T}_\alpha$. We show by induction that $|\mathcal{T}_\alpha| = 2^{\aleph_0}$ for all $\alpha < \aleph_1$. For $\alpha = 0$, it is just the previous point. If $|\mathcal{T}_\alpha| = 2^{\aleph_0}$, then:

   $$2^{\aleph_0} = |\mathcal{T}_\alpha| \leq |\mathcal{T}_{\alpha+1}| \leq 2^{\aleph_0} + 3 \cdot (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}.$$

   If $\lambda < \aleph_1$ is limit, then:

   $$2^{\aleph_0} = |\mathcal{T}_0| \leq |\mathcal{T}_\lambda| = \left|\bigcup_{\alpha<\lambda} \mathcal{T}_\alpha\right| \leq |\lambda| \cdot 2^{\aleph_0} = \aleph_0 \cdot 2^{\aleph_0} = 2^{\aleph_0}.$$

   It follows that:

   $$2^{\aleph_0} \leq |\mathcal{B}| = \left|\bigcup_{\alpha<\aleph_1} \mathcal{T}_\alpha\right| \leq \aleph_1 \cdot 2^{\aleph_0} = 2^{\aleph_0}.$$

6. Denote by $\mathcal{C}$ the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. On one hand, the function $l : \mathbb{R} \to \mathcal{C}, p \mapsto (x \mapsto p \cdot x)$ is injective and therefore $2^{\aleph_0} \leq |\mathcal{C}|$. On the other hand, by the density of the rationals in the reals, each function of $\mathcal{C}$ is uniquely determined by its values on $\mathbb{Q}$. Thus, the function $\mathcal{C} \mapsto \mathbb{R}^{\mathbb{Q}}, f \mapsto f{\restriction}_{\mathbb{Q}}$ is injective and therefore $|\mathcal{C}| \leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}$.

7. Let f be the pointwise limit of a succession $(f_n)_{n\in\omega}$ of Borel functions. We prove that the $f^{-1}(U)$ is Borel for each open subset U of $\mathbb{R}$. First notice that it is enough to prove it for basic open sets $U =]a,b[$, since each open set is a countable union of basic open sets, and union and preimage commute.

   Let $a < b$. Then $]a,b[= \bigcup_{m\in\omega, m>1} \left[a + \frac{b-a}{m}, b - \frac{b-a}{m}\right] = \bigcup_{m\in\omega,m>1} \left]a + \frac{b-a}{m}, b - \frac{b-a}{m}\right[$. We claim that:
$$f^{-1}(]a,b[) = \bigcup_{m\in\omega,m>1} \bigcup_{n\in\omega} \bigcap_{k\geq n} f_k^{-1}\left(\left]a + \frac{b-a}{m}, b - \frac{b-a}{m}\right[\right).$$

   Let $x \in \mathbb{R}$ and V be an open set, then we have the following:
$$\begin{aligned} f(x) \in V \quad &\longleftrightarrow \quad \lim_{n\to\infty} f_n(x) \in V \\ &\longrightarrow \quad \exists n\, \forall k \geq n\ f_k(x) \in V \\ &\longrightarrow \quad f(x) \in \overline{V}. \end{aligned}$$

   Therefore:
$$\begin{aligned} f^{-1}(]a,b[) = f^{-1}\left(\bigcup_{m\in\omega,m>1} \left]a + \frac{b-a}{m}, b - \frac{b-a}{m}\right[\right) &= \bigcup_{m\in\omega,m>1} f^{-1}\left(\left]a + \frac{b-a}{m}, b - \frac{b-a}{m}\right[\right) \subseteq \\ \subseteq \bigcup_{m\in\omega,m>1} \bigcup_{n\in\omega} \bigcap_{k\geq n} f_k^{-1}\left(\left]a + \frac{b-a}{m}, b - \frac{b-a}{m}\right[\right) &\subseteq \bigcup_{m\in\omega,m>1} f^{-1}\left(\left[a + \frac{b-a}{m}, b - \frac{b-a}{m}\right]\right) = \\ = f^{-1}\left(\bigcup_{m\in\omega,m>1} \left[a + \frac{b-a}{m}, b - \frac{b-a}{m}\right]\right) &= f^{-1}(]a,b[), \end{aligned}$$

   which proves the claim.

   Since each f_k is Borel and $\left]a + \frac{b-a}{m}, b - \frac{b-a}{m}\right[$ is open, $f_k^{-1}\left(\left]a + \frac{b-a}{m}, b - \frac{b-a}{m}\right[\right)$ is Borel. Therefore $f^{-1}(]a,b[)$ is Borel as it is a countable union of countable intersections of Borel sets.

8. Denote by $l_p : (\mathbb{R}^\mathbb{R})^{\aleph_0} \to \mathbb{R}^\mathbb{R}$ the operation defined by:
$$l_p((f_n)_{n<\omega}) = \begin{cases} x \mapsto \lim_{n\to\infty} f_n(x) & \text{if } (f_n(x))_{n\in\omega} \text{ converges for all } x \in \mathbb{R}, \\ f_0 & \text{if not.} \end{cases}$$

   We have that $\mathfrak{B}$ is the closure of $\mathcal{C}$ under the operation l_p. As before we show that:
$$2^{\aleph_0} \leq |\mathfrak{B}| = \left|\bigcup_{\alpha<\omega_1} \mathcal{C}_\alpha\right| \leq \aleph_1 \cdot 2^{\aleph_0} = 2^{\aleph_0}.$$

Solution of exercise 2:

1.1 By the condition **iii)** we have that if $X \subseteq Y$ are two bounded sets of reals $m(Y) = m((Y \setminus X) \cup X) = m(Y \setminus X) + m(X) \geq m(X)$.

1.2 Each solution to Lebesgue's measure problem is determined by its values on the subsets of the unit interval $[0,1]$:

   Let B be a bounded set of reals. There exists an integer $k \in \mathbb{Z}$ and a natural number $n > 0$ such that $B \subseteq [k, k+n]$. Let $B_i = B \cap [k+i, k+i+1[$ for each $i = 0, \ldots, n-1$. Then B is the disjoint union of the B_i's and, for each i, $B_i - (k+i) \subseteq [0,1]$. Thus by the conditions **ii)** and **iii)** it follows that $m(B) = \sum_{i=0}^{n-1} m(B_i) = \sum_{i=0}^{n-1} m(B_i) - (k+1)$.

2.1-6

Theorem (Giuseppe Vitali, 1907)**.** *Assuming the axiom of choice, there does not exist of function solving Lebesgue's measure problem.*

Proof. Suppose that there exists of a function m satisfying the conditions **i)**, **ii)** and **iii)** above. Consider the following equivalence relation on the real numbers: $x \sim y$ if and only if $x - y$ is rational. The intersection between the equivalence class of a real number x and the interval $[0,1]$ is equal to the set $\{x+r \in [0,1] \mid r \text{ is rational}\}$. By the axiom of choice, there exists a set $\mathcal{V}$ containing exactly an element of the intersection of each equivalence class with the interval $[0,1]$. We call this Vitali's set. By definition $\mathcal{V} \subseteq [0,1]$, so $\mathcal{V}$ is a bounded set of real numbers. For each rational number $r \in [-1,1]$, denote by $\mathcal{V}+r = \{v+r \mid v \in \mathcal{V}\}$ the translation by r of Vitali's set. Notice that for two distinct rationals $r, s \in [-1,1]$, $(\mathcal{V}+r) \cap (\mathcal{V}+s) = \emptyset$. Indeed, if $c \in (\mathcal{V}+r) \cap (\mathcal{V}+s)$, then there exist $v_1, v_2 \in \mathcal{V}$ such that $v_1 - v_2 = s - r$, so $v_1 \sim v_2$ and, by definition of $\mathcal{V}$, necessarily $v_1 = v_2$, and therefore $r = s$. We then consider the following set:

$$X = \bigcup_{r \in [-1,1] \cap \mathbb{Q}} \mathcal{V} + r.$$

We have that $[0,1] \subseteq X \subseteq [-1,2]$. It follows by monotony of m that:

$$m([0,1]) \leq m(X) \leq m([-1,2]).$$

Now, by **iii)**, $m(X) = \sum_{r \in [-1,1] \cap \mathbb{Q}} m(\mathcal{V}+r)$ and by **ii)**, we have $m(\mathcal{V}+r) = m(\mathcal{V})$ for every rational $r \in [-1,1]$. Thus, it is not possible that $m(\mathcal{V})$ is strictly positive since then $m([-1,2])$ would be larger than all natural numbers. The fact that $m(\mathcal{V})$ is 0 implies that $m([0,1]) = 0$ and this contradicts the condition **i)** by the point 1.2 above. □

3.1 Checking that points **a)** and **c)** hold is immediate, and the fact that $m_x(\{x\}) = 1$ contradicts **b)**.

4.1 *Let m be a measure on a set S. Show that every family $T \subseteq \mathcal{P}(S)$ of pairwise disjoint sets contains at most a countably many sets of positive measure.*

Suppose towards contradiction that there exists an uncountable family $T \subseteq \mathcal{P}(S)$ of pairwise disjoint sets of strictly positive measure. Then, there exists some $N \in \omega$, $N > 0$ such that $T_N = \{X \in T \mid m(X) > \frac{1}{N}\}$ is uncountable. This follows from the fact that a countable union of countable sets is countable (by the axiom of countable choice) and from the fact that $T = \bigcup_{n>0} T_n$. Thus, there exist $X_1, \ldots, X_N \in T_N$ which are distinct and verify the σ-additivity of m:

$$m\Big(\bigcup_{k=1}^{N} X_k\Big) = \sum_{k=1}^{N} m(X_k) > 1.$$

This contradicts, by monotony of m, the fact that $m(S) = 1$.

Lemma (facultative)**.** *If κ is the smallest cardinal such that there exists a measure on κ, then any measure on κ is κ-additive.*

Proof. Suppose towards contradiction that κ is the smallest cardinal which admits a measure and that there exists a measure m on κ which is not κ-additive. There then exists an ordinal $\gamma < \kappa$ and a collection $\langle X_\alpha \mid \alpha < \gamma \rangle$ of pairwise disjoint subsets of κ such that $m\big(\bigcup_{\alpha<\gamma} X_\alpha\big) \neq \sum_{\alpha<\gamma} m(X_\alpha)$. Since m is $\aleph_1$-additive by definition, necessarily $\gamma \geq \aleph_1$. Moreover, at most a countable number of X_α's have positive measure by 3.1 above. Thus by $\aleph_1$-additivity of m, we have:

$$0 = \sum_{\alpha<\gamma} m(X_\alpha) - \sum_{\substack{\beta \text{ such that} \\ m(X_\beta)>0}} m(X_\beta) = \sum_{\alpha<\gamma} m(X_\alpha) - m\Big(\bigcup_{\substack{\beta \text{ such that} \\ m(X_\beta)>0}} X_\beta\Big)$$

$$\neq m\Big(\bigcup_{\alpha<\gamma} X_\alpha\Big) - m\Big(\bigcup_{\substack{\beta \text{ such that} \\ m(X_\beta)>0}} X_\beta\Big) = m\Big(\bigcup_{\substack{\alpha \text{ such that} \\ m(X_\alpha)=0}} X_\alpha\Big).$$

Thus, for the cardinal $\lambda = |\gamma|$, which verifies $\aleph_1 \leq \lambda < \kappa$, we have a collection $\{X_\alpha \mid \alpha < \lambda\}$ of pairwise disjoint subsets of κ verifying, on one hand, that $m(X_\alpha) = 0$ for all $\alpha < \lambda$ and, on the other, that $m(\bigcup_{\alpha<\lambda} X_\alpha) > 0$. Denote by $r = m(\bigcup_{\alpha<\lambda} X_\alpha)$ and define a measure $\overline{m}$ on λ by letting:

$$\overline{m}(X) = \frac{m\big(\bigcup_{\alpha\in X} X_\alpha\big)}{r},$$

for all $X \subseteq \lambda$. Indeed, we have $\overline{m}(\lambda) = \frac{r}{r} = 1$, $\overline{m}(\{\alpha\}) = \frac{m(X_\alpha)}{r} = 0$ for all $\alpha < \lambda$ and the $\aleph_1$-additivity of $\overline{m}$ follows from the $\aleph_1$-additivity of m. Therefore, $\overline{m}$ contradicts the minimality of κ. □

Let κ be a real-valued measurable cardinal and m a κ-additive measure on κ.

5.1 *For all $X \subseteq \kappa$ with $|X| < \kappa$, we have $m(X) = 0$:* Indeed, $m(X) = m\Big(\bigcup_{x\in X}\{x\}\Big) = \sum_{x\in X} m(\{x\}) = 0$ by κ-additivity of m.

5.2 *κ is regular.* The previous point assures us, in particular, that $m(\beta) = 0$ for all $\beta < \kappa$. Let $\alpha < \kappa$ and $f : \alpha \to \kappa$ be a function. We have $|f[\alpha]| \leq \alpha < \kappa$ and thus:

$$m(\sup f[\alpha]) = m\Big(\bigcup f[\alpha]\Big) \leq \sum_{\beta\in f[\alpha]} m(\beta) = 0.$$

Since $m(\kappa) = 1$, it must be that $\sup f[\alpha] < \kappa$ and therefore f is not cofinal.

We have the following theorems.

Theorem (Ulam, 1930)**.** *If κ is real-valued measurable, then κ is weakly inaccessible.*

Theorem (Ulam, 1930)**.** *If there exists a κ-additive atomless measure on κ, then $\kappa \leq 2^{\aleph_0}$.*

The existence of a real-valued measurable cardinal with an atomless measure thus strongly contradicts the continuum hypothesis. Indeed, such a cardinal κ would be weekly inaccessible, that is a regular limit cardinal, but less or equal to $2^{\aleph_0}$. Since κ is limit:

$$\kappa = \bigcup\{\gamma \in \kappa \mid \gamma \text{ cardinal}\},$$

so, being that $\mathrm{cof}(\kappa) = \kappa$, it must be that $|\{\gamma \in \kappa \mid \gamma \text{ cardinal}\}| \geq \kappa$. There would thus exist at least κ infinite cardinals below κ, and therefore below $2^{\aleph_0}$.

Let κ be a cardinal and m a κ-additive measure with an atom $A \subseteq \kappa$.

6.1 Straightforward.

6.2 The only delicate point is the κ-completeness of U_μ. To prove it, we show the following lemma:

Lemma. *An ultrafilter U on a set S is λ-complete if and only if for all $\gamma < \lambda$ and every family $\langle X_\alpha \mid \alpha < \gamma\rangle$ of subsets of S, if $\bigcup_{\alpha<\gamma} X_\alpha \in U$, then there exists $\alpha < \gamma$ such that $X_\alpha \in U$.*

Proof. By contraposition, suppose that there exists $\gamma < \lambda$ and a family $\langle X_\alpha \mid \alpha < \gamma\rangle$ of subsets of S such that $\bigcup_{\alpha<\gamma} X_\alpha \in U$ but, for all $\alpha < \gamma$, $X_\alpha \notin U$. Thus $S - X_\alpha \in U$ for all α, and:

$$S - \bigcap_{\alpha<\gamma}(S - X_\alpha) = \bigcup_{\alpha<\gamma} X_\alpha \in U.$$

Therefore $\bigcap_{\alpha<\gamma}(S - X_\alpha) \notin U$ and therefore U is not λ-complete.

On the other hand, again by contraposition, suppose that there exists $\gamma < \lambda$ and a family $\langle X_\alpha \mid \alpha < \gamma\rangle$ of elements of U such that $\bigcap_{\alpha<\gamma} X_\alpha \notin U$. In this case, $\bigcup_{\alpha<\gamma}(S - X_\alpha) = S - \bigcap_{\alpha<\gamma} X_\alpha \in U$, even if $S - X_\alpha \notin U$ for all $\alpha < \gamma$. □

Let us now check that the κ-additivity of μ implies the κ-completeness of U_μ. Let $\gamma < \kappa$ and consider a family $\langle X_\alpha \mid \alpha < \gamma \rangle$ of subsets of κ such that $\bigcup_{\alpha<\gamma} X_\alpha \in U_\mu$. Then if $X_\alpha \notin U_\mu$ for all α, i.e. $\mu(X_\alpha) = 0$ the κ-additivity of μ would imply that $\mu\big(\bigcup_{\alpha<\gamma} X_\alpha\big) \leq \sum_{\alpha<\gamma} \mu(X_\alpha) = 0$, contradicting the fact that $\bigcup_{\alpha<\gamma} X_\alpha \in U_\mu$.

Definition. *A cardinal $\kappa > \omega$ is* measurable *if there exists a non-principal κ-complete ultrafilter on κ.*

7.1

Theorem (Ulam-Tarski, 1930)**.** *Any measurable cardinal κ is strongly inaccessible.*

Proof. Let U be a κ-complete ultrafilter on a cardinal $\kappa > \omega$. It is straightforward to check that the measure on κ defined by $m_U(X) = 1$, if $X \in U$, and $m_U(X) = 0$, if not, is κ-additive. The regularity of κ follows therefore from point 5.2 above.

To prove that for all $\lambda < \kappa$, $2^\lambda < \kappa$, suppose towards contradiction that for a cardinal $\lambda < \kappa$, there is an injective function $f : \kappa \to {}^\lambda 2$. Consider, for all $\alpha < \lambda$, the sets $X_\alpha = \{\xi \in \kappa \mid f(\xi)(\alpha) = i_\alpha\}$, where $i_\alpha \in \{0, 1\}$ is such that $X_\alpha \in U$. By κ-completeness of U, we have $X = \bigcap_{\alpha<\lambda} X_\alpha \in U$. Therefore, if $\eta, \xi \in X$, then for all $\alpha \in \lambda$, $f(\xi)(\alpha) = i_\alpha = f(\eta)(\alpha)$, i.e. $f(\xi) = f(\eta)$, and therefore $\xi = \eta$ by injectivity of f. This would imply that X has at most one element, contradicting the fact that $X \in U$. □